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Recurrence rates for observations of flows

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Abstract. We study Poincaré recurrence for flows and observations of flows. For Anosov flow, we prove that the recurrence rates are linked to the local dimension of the invariant measure. More generally, we give for the recurrence rates for the observations an upper bound depending on the push-forward measure. When the flow is metrically isomorphic to a suspension flow for which the dynamic on the base is rapidly mixing, we prove the existence of a lower bound for the recurrence rates for the observations. We apply these results to the geodesic flow and we compute the recurrence rates for a particular observation of the geodesic flow, i.e. the projection on the manifold.

1. *Introduction*

1.1. *Poincaré recurrence.*

One of the fundamental theorems at the origin of dynamical systems and ergodic theory is the Poincaré recurrence theorem. It states that in a dynamical system almost every orbit returns as closely as you wish to the initial point. The time needed for a point to come back, called the return time, has been widely studied in the last few years for discrete dynamical systems (e.g. [**1, 6, 12, 13, 18, 24**]).

A noteworthy work on the quantitative study of Poincaré recurrence is the paper by Boshernitzan [**9**]. In this work, Boshernitzan introduced the idea of studying Poincaré recurrence for observations of dynamical systems. Indeed, he proved that for a dynamical system (X, T, μ) and an observation f from X to a metric space (Y, d), whenever the α-dimensional Hausdorff measure is σ-finite on Y, we have

$$\liminf_{n\to\infty} n^{1/\alpha} d(f(x), f(T^n x)) < \infty \quad \text{for } \mu\text{-almost every } x. \tag{1}$$

Following this idea and the work in [**6, 23**], recurrence rates for observations have been linked to the pointwise dimensions of the push-forward measure [**21**]. An example of application of observations of dynamical systems is the study of recurrence for random dynamical systems [**17**].

It is natural to wonder if these results can be extended to continuous time, and if one can obtain quantitative results for recurrence of flows and more generally for observations of flows. Barreira and Saussol [**6**] proved that for a suspension flow over an Anosov diffeomorphism such that the invariant measure is an equilibrium state of a Hölder

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potential, the return time of ν-almost every point y in the ball $B(y, r)$ behaves like $r^{-\dim_H \nu+1}$ when r goes to zero (similar results have been proved for the hitting time of Lorenz-like flows [**14**]).

Pène and Saussol [**19**] studied the billiard flow in a plane with a periodic configuration of scatterers. They proved that, almost everywhere, the return time of a point (p, v) in the ball $B((p, v), r)$ is of the order $\exp(1/r^2)$ and that the return time of the position of a point in $B(p, r)$, i.e. the return time of the projection of the flow on the billiard, is of the order $\exp(1/r)$ almost everywhere.

Following these works and the ideas of Boshernitzan, we study the recurrence rates for flows and observations of flows, in particular we obtain some results for geodesic flow.

1.2. *Statement of the principal results.* Let M be a compact Riemannian manifold and d the Riemannian metric. Let Ψ be a flow on M. Let ν be a probability measure on M, invariant for the flow Ψ. We introduce the notion of return time and recurrence rates for flows.

Definition 1.1. We define for $x \in M$ the return time of the flow Ψ as

$$\tau_r^{\Psi}(x) := \inf\{t > \eta_r(x) : \Psi_t(x) \in B(x, r)\},$$

where $B(x, r)$ is the ball centred in x and of radius r and $\eta_r(x)$ is the first escape time of the ball $B(x, r)$, i.e. $\eta_r(x) = \inf\{t > 0, \Psi_t x \notin B(x, r)\}$. We also define the lower and upper recurrence rates:

$$\underline{R}^{\Psi}(x) := \liminf_{r\to 0} \frac{\log \tau_r^{\Psi}(x)}{-\log r} \quad \text{and} \quad \overline{R}^{\Psi}(x) := \limsup_{r\to 0} \frac{\log \tau_r^{\Psi}(x)}{-\log r}.$$

We will show that these recurrence rates are linked to the local dimension of the invariant measure. We recall that *the lower and upper pointwise or local dimension* of a Borel probability measure μ on a metric space X at a point $x \in X$ are defined by

$$\underline{d}_{\mu}(x) = \liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_{\mu}(x) = \limsup_{r\to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

First, we will prove a theorem satisfied for any flow.

THEOREM 1.2. *Let Ψ be a differentiable flow on M and ν an invariant probability measure for Ψ. For ν-almost every $x \in M$ which is not a fixed point,*

$$\underline{R}^{\Psi}(x) \le \underline{d}_{\nu}(x) - 1 \quad \textit{and} \quad \overline{R}^{\Psi}(x) \le \overline{d}_{\nu}(x) - 1.$$

To obtain an equality between recurrence rates and dimensions, we need more assumptions on the system.

THEOREM 1.3. *Let Ψ be an Anosov flow on M. If ν is an equilibrium state of a Hölder potential, then*

$$\underline{R}^{\Psi}(x) = \underline{d}_{\nu}(x) - 1 \quad \textit{and} \quad \overline{R}^{\Psi}(x) = \overline{d}_{\nu}(x) - 1$$

for ν-almost every $x \in M$.

The existence of the local dimension of the invariant measure for hyperbolic flows is still an open question. Indeed, even if for hyperbolic diffeomorphisms the existence of the local dimension has been proved [**4, 16**] and there exists an explicit formula to compute it [**7**], for hyperbolic flows, the existence has been proved only in the conformal case [**8, 20**].

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For example, we can apply the previous theorem to the geodesic flow on a smooth manifold with stricly negative curvature. Since the geodesic flow is defined on the unit tangent bundle T^1M, we can also consider a particular observation of this flow: the position on the manifold M. Let Π be the canonical projection:

$$\Pi : T^1M \longrightarrow M,$$
$$(p, v) \longmapsto p.$$

We study the return time for the canonical projection on the manifold M:

$$\tau_r^{\Psi,\Pi}(p, v) := \inf\{t > r : \Pi(\Psi_t(p, v)) \in B(p, r)\}.$$

Since Ψ is the geodesic flow on T^1M, the first escape time of the projection of the flow on the manifold of the ball $B(p, r)$ is equal to r for r small enough. We define the recurrence rates for the canonical projection:

$$\underline{R}^{\Psi,\Pi}(p, v) := \liminf_{r\to 0} \frac{\log \tau_r^{\Psi,\Pi}(p, v)}{-\log r} \quad \text{and} \quad \overline{R}^{\Psi,\Pi}(p, v) := \limsup_{r\to 0} \frac{\log \tau_r^{\Psi,\Pi}(p, v)}{-\log r}.$$

THEOREM 1.4. *Let Ψ be the geodesic flow defined on T^1M and ν an invariant probability measure for Ψ. Then, for ν-almost every $(p, v) \in T^1M$,*

$$\underline{R}^{\Psi,\Pi}(p, v) \leq \underline{d}_{\Pi_*\nu}(p) - 1 \quad \textit{and} \quad \overline{R}^{\Psi,\Pi}(p, v) \leq \overline{d}_{\Pi_*\nu}(p) - 1.$$

Moreover, if M has a strictly negative curvature and if ν is an equilibrium state of a Hölder potential, then

$$\underline{R}^{\Psi,\Pi}(p, v) = \underline{d}_{\Pi_*\nu}(p) - 1 \quad \textit{and} \quad \overline{R}^{\Psi,\Pi} = \overline{d}_{\Pi_*\nu}(p) - 1$$

for ν-almost every $(p, v) \in T^1M$ non-multiple such that $\underline{d}_{\Pi_*\nu}(p) > 1$.

Since the geodesic flow preserves the Lebesgue measure on T^1M, we can apply Theorems 1.3 and 1.4 to obtain the following noteworthy result.

COROLLARY 1.5. *Let M be an n-dimensional manifold with strictly negative curvature. Let Ψ be the geodesic flow defined on T^1M. Then, for Lebesgue-almost every $(p, v) \in T^1M$,*

$$R^{\Psi}(p, v) = 2n - 2$$

and

$$R^{\Psi,\Pi}(p, v) = n - 1.$$

The structure of the paper is as follows. First, we introduce the notion of escape function and give an upper bound for the recurrence rate for observations in a general setting.

Then, in §3, using suspension flows, we obtain a lower bound of the recurrence rate for the observation for flows presenting some hyperbolic behaviour.

In §4, we use these results to compute the recurrence rates of the geodesic flow and for a particular observation of the geodesic flow, the projection on the manifold.

Then, in §5, we give some simple but essential examples of observations to understand the concepts of escape function and projection dimension.

Finally, in the last two sections, we prove the principal theorems.

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2. *Recurrence for observations of flows*

Let M be a Riemannian manifold and d the Riemannian metric. Let $\mathcal{B}(M)$ be the Borel σ-algebra of M and let ν be a probability measure on M. Let $\Psi = \{\Psi_t\}_{t\in\mathbb{R}}$ be a measurable flow on M. We recall that the measure ν is invariant for Ψ if

$$\nu(\Psi_t^{-1}(A)) = \nu(A) \quad \text{for all } t\in\mathbb{R}, \text{ for all } A \in \mathcal{B}(M).$$

Let $N \in \mathbb{N}$. As in [**21**], we are interested in observations of dynamical systems. We consider an observation $f : M \to \mathbb{R}^N$ and we study the image by f of the orbits of Ψ. In particular, we are going to study Poincaré recurrence for the observation of the flow Ψ. First of all, we need to introduce the notion of *escape function*.

Definition 2.1. Let ρ be a function defined from $[0, +\infty) \times M$ to $[0, +\infty)$:

$$\begin{aligned} \rho : [0, +\infty) \times M &\longrightarrow [0, +\infty) \\ (r, z) &\longmapsto \rho_r(z). \end{aligned}$$

ρ is called an escape function if for all $1 > \xi_1 > 0$, for all $\xi_2 > 1$, for all $\varepsilon > 0$, and for $f_\star\nu$-almost every $z_1 \in \mathbb{R}^N$, $\zeta > 0$ exists such that if $r < \zeta$, then for $f_\star\nu$-almost every $z_2 \in B(z_1, \min\{(1 - \xi_1)r, (\xi_2 - 1)r\})$, we have

$$r^\varepsilon \rho_{\xi_1 r}(z_2) \le \rho_r(z_1) \le \rho_{\xi_2 r}(z_2) r^{-\varepsilon}.$$

To understand this concept, we give some examples of escape functions and we also refer to §5.

(1) For $\alpha \in [0, 1]$,

$$\rho_r(z) = r^\alpha |\log r|.$$

(2) $$\rho_r(z) = \frac{1}{f_\star\nu(B(z, r))} \int_{f^{-1}B(z,r)} \inf\{t > 0, \Psi_t y \notin f^{-1}B(z, r)\}\, d\nu(y).$$

(3) $$\rho_r(z) = \frac{|\log r|}{f_\star\nu(B(z, r))} \int_{f^{-1}B(z,r)} \inf\{t > 0, \Psi_t y \notin f^{-1}B(z, r)\}\, d\nu(y).$$

(4) $$\rho_r(z) := \operatorname*{ess-sup}_{y\in f^{-1}B(z,r)} \{\inf\{t > 0 : \Psi_t y \notin f^{-1}B(z, r)\}\}.$$

We now introduce the notion of return time and recurrence rates for the observation of the flow.

Definition 2.2. Let $f : M \to \mathbb{R}^N$ be a measurable function and ρ an escape function; we define for $x \in M$ the return time for the observation of the flow Ψ with respect to ρ,

$$\tau_{r,\rho}^{\Psi,f}(x) := \inf\{t > \rho_r(f(x)) : f(\Psi_t(x)) \in B(f(x), r)\},$$

where $B(f(x), r)$ denotes the ball centred in $f(x)$ and of radius r. We also define the lower and upper recurrence rates for the observation of the flow Ψ with respect to ρ:

$$\underline{R}_\rho^{\Psi,f}(x) := \liminf_{r\to 0} \frac{\log \tau_{r,\rho}^{\Psi,f}(x)}{-\log r} \quad \text{and} \quad \overline{R}_\rho^{\Psi,f}(x) := \limsup_{r\to 0} \frac{\log \tau_{r,\rho}^{\Psi,f}(x)}{-\log r}.$$

Let $p \in \mathbb{N}$. We also introduce the p-non-instantaneous return time for the observation of the flow Ψ:

$$\tau_{r,p}^{\Psi,f,\star}(x) = \inf\{t > p : f(\Psi_t(x)) \in B(f(x), r)\},$$

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and the lower and upper non-instantaneous recurrence rates for the observation of the flow Ψ:

$$\underline{R}_{\star}^{\Psi,f}(x) = \lim_{p\to+\infty} \liminf_{r\to 0} \frac{\log \tau_{r,p}^{\Psi,f,\star}(x)}{-\log r}$$

and

$$\overline{R}_{\star}^{\Psi,f}(x) = \lim_{p\to+\infty} \limsup_{r\to 0} \frac{\log \tau_{r,p}^{\Psi,f,\star}(x)}{-\log r}.$$

As explained in [**17**], it can be useful to work with non-instantaneous return times since, with the observation, there can be a loss of information. We note that these recurrence rates can be equal.

PROPOSITION 2.3. *Let ρ be an escape function. If for $x \in M$, $\underline{R}_{\rho}^{\Psi,f}(x) > 0$, then for any other escape function $\tilde{\rho}$ such that $\tilde{\rho}_r(f(x)) \geq \rho_r(f(x))$ for all r small enough and such that $\underline{\lim}_{r\to 0}(\log \tilde{\rho}_r(f(x))/\log r) \geq 0$, we have*

$$\underline{R}_{\rho}^{\Psi,f}(x) = \underline{R}_{\tilde{\rho}}^{\Psi,f}(x) = \underline{R}_{\star}^{\Psi,f}(x)$$

and

$$\overline{R}_{\rho}^{\Psi,f}(x) = \overline{R}_{\tilde{\rho}}^{\Psi,f}(x) = \overline{R}_{\star}^{\Psi,f}(x).$$

Proof. We just need to remark that if for $x \in M$, $\underline{R}_{\rho}^{\Psi,f}(x) > 0$, then $a > 0$ exists such that for all $r > 0$ small enough, we have

$$\tau_{r,\rho}^{\Psi,f}(x) \geq r^{-a}.$$ □

For discrete dynamical systems, it is proved in [**6**] that recurrence rates are linked to the pointwise dimensions of the invariant measure, and in [**21**] that recurrence rates for the observation are linked to the pointwise dimension of the push-forward measure. For flows, the recurrence rates for the observation are linked to the pointwise dimension of the push-forward measure, but intrinsically, since theses rates depend also on the escape function. The idea is to find an optimal escape function, i.e. an escape function such that almost everywhere the escape time of the observation of the flow from $B(f(x), r)$ is smaller than the escape function when the radius is small enough and such that $\rho_r(f(x))$ is the smallest possible when $r \to 0$. So, the first theorem on recurrence for observations of flows is the following (it will be proved in §6).

THEOREM 2.4. *Let Ψ be a flow on M and ν a probability measure Ψ-invariant. Let f be a measurable observation from M to $\mathbb{R}^N$ and let ρ be an escape function. Then, for ν-almost every $x \in M$,*

$$\underline{R}_{\rho}^{\Psi,f}(x) \leq \liminf_{r\to 0}\left(\frac{\log f_{\star}\nu(B(f(x), r))}{\log r} - \frac{\log \rho_r(f(x))}{\log r}\right)$$

and

$$\overline{R}_{\rho}^{\Psi,f}(x) \leq \limsup_{r\to 0}\left(\frac{\log f_{\star}\nu(B(f(x), r))}{\log r} - \frac{\log \rho_r(f(x))}{\log r}\right).$$

We emphasize that this result cannot be obtained using the time-1 map, i.e. $Tx = \Psi_1(x)$, which gives non-optimal results. Indeed, using the results of [**21**] with the time-1 map,

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we obtain that for ν-almost every $x \in M$,

$$\underline{R}_\rho^{\Psi,f}(x) \le \underline{d}_{f_\star\nu}(f(x)) \quad \text{and} \quad \overline{R}_\rho^{\Psi,f}(x) \le \overline{d}_{f_\star\nu}(f(x)).$$

This result is optimal when

$$\lim_{r\to 0} \frac{\log \rho_r(f(x))}{\log r} = 0$$

but it is a degenerate case, as we will see in §5.

We notice that when the observation is the identity, we obtain Theorem 1.2 for points which are not fixed points ($t \in \mathbb{R}$ exists such that $\Psi_t x \neq x$).

Proof of Theorem 1.2. Using Theorem 2.4 with f the observation identity,

$$f : M \longrightarrow M, \qquad x \longmapsto x$$

and choosing the escape function ρ as follows,

$$\rho_r : M \longrightarrow \mathbb{R}, \qquad x \longmapsto \rho_r(x) = r|\log r|$$

we obtain that

$$\underline{R}_\rho^{\Psi,f}(x) \le \underline{d}_\nu(x) - 1 \quad \text{and} \quad \overline{R}_\rho^{\Psi,f}(x) \le \overline{d}_\nu(x) - 1.$$

We still need to prove that $\underline{R}^\Psi(x) \le \underline{R}_\rho^{\Psi,f}(x)$ and that $\overline{R}^\Psi(x) \le \overline{R}_\rho^{\Psi,f}(x)$. For this, we only need to prove that for almost every $x \in M$ and for r small enough we have $\tau_r^\Psi(x) \le \tau_{r,\rho}^\Psi(x)$. By the flow box theorem [**3**], and since the flow is differentiable, around a non-fixed point there exists a neighborhood U, a constant γ and a time T such that for all $z \in U$ and for all $0 < t < T$, $d(z, \Psi_t z) \ge \gamma t$. Then, for all $x \in M$ which are not fixed points, $r_1(x) > 0$ exists such that for all $0 < r < r_1$, $\inf\{t > 0, \Psi_t x \notin B(x, r)\} \le r|\log r|$, and then $\tau_r^\Psi(x) \le \tau_{r,\rho}^\Psi(x)$. □

As in [**6, 21, 23**], to obtain a lower bound for recurrence rates, we need more assumptions on the system. For flows, an assumption on the speed of decay of correlations will not be optimal. Indeed, it is possible to construct Axiom A flow with an arbitrarily slow decay of correlations (e.g. [**22**]). This is why we will use suspension flow and the assumption on the speed of mixing will be for the dynamic on the base.

3. *Recurrence for observations via suspension flows*

From now on, we suppose that the manifold M is compact. To get a lower estimate of the recurrence rate, we use the suspension or special flow.

Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure-preserving system, i.e. $\mathcal{A}$ is a σ-algebra, μ is a measure on $(X, \mathcal{A})$ with $\mu(X) = 1$, μ is invariant for T (i.e. $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{A}$), where $T : X \to X$ and d is a metric.

Let $\phi : X \to (0, +\infty)$ be a measurable and integrable function. We define the space under the function ϕ

$$Y := \{(u, s) \in X \times \mathbb{R} : 0 \le s \le \phi(u)\},$$

where $(u, \phi(u))$ and $(Tu, 0)$ are identified for all $u \in X$. The *suspension flow* or the *special flow* over T with height function ϕ is the flow Φ which acts on Y by the following

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transformation:

$$\Phi_t(u, s) = (u, s + t) \quad \text{for all } (u, s, t) \in X \times \mathbb{R}^+ \times \mathbb{R}^+.$$

The metric on Y is the Bowen–Walters distance (e.g. [**11**]). First, we recall the definition of the Bowen–Walters distance d_1 on Y when $\phi(x) = 1$ for every $x \in X$. Let $x, y \in X$ and $t \in [0, 1]$; the length of the horizontal segment $[(x, t), (y, t)]$ is defined by

$$\alpha_h((x, t), (y, t)) = (1 - t)d(x, y) + td(Tx, Ty).$$

Let $(x, t), (y, s) \in Y$ be on the same orbit; the lenght of the vertical segment $[(x, t), (y, s)]$ is defined by

$$\alpha_v((x, t), (y, s)) = \inf\{|r| : \Phi_r(x, t) = (y, s) \text{ and } r \in \mathbb{R}\}.$$

Let $(x, t), (y, s) \in Y$; the distance $d_1((x, t), (y, s))$ is defined as the infimum of the lenghts of paths between (x, t) and (y, s) composed of a finite number of horizontal and vertical segments. When ϕ is arbitrary, the Bowen–Walters distance on Y is given by

$$d((x, t), (y, s)) = d_1\left(\left(x, \frac{t}{\phi(x)}\right), \left(y, \frac{s}{\phi(y)}\right)\right).$$

For more details on the Bowen–Walters distance, see [**5**, Appendix A].

We recall that the measure ν_μ defined by $\mu \otimes \text{Leb}$ on Y and normalized is invariant for the flow Φ. Generally, flow and suspension flow are linked.

THEOREM 3.1. [**2**] *Any flow Ψ without fixed points is metrically isomorphic to a special flow Φ.*

This means that for a flow Ψ on M with invariant measure ν, there exists a suspension flow Φ over T with height function ϕ and an invariant measure ν_μ such that there exists a set M' such that $\nu(M') = 1$, a set Y' such that $\nu_\mu(Y') = 1$ and a function $g : Y \to M$ which is one-to-one from Y' to M' such that for all $t \geq 0$,

$$\Psi_t \circ g = g \circ \Phi_t \quad \text{and} \quad \Psi_t \circ g^{-1} = g^{-1} \circ \Phi_t,$$

and such that for all measurable subsets $A \subset M$ and all measurable subsets $B \subset Y$,

$$\nu(A) = \nu_\mu(g^{-1}A) \quad \text{and} \quad \nu_\mu(B) = \nu(g(B)).$$

We emphasize that for hyperbolic flows, we can choose the metric on X such that the function g is Lipschitz [**10**].

In the study of recurrence for observations of suspension flows, it appears that the recurrence rates are bounded from below by some new quantities. This is why we introduce the definition of the local projection dimension of the associated suspension flow.

Definition 3.2. Let Ψ be a flow metrically isomorphic to a suspension flow Φ over $(X, \mathcal{A}, T, \mu)$ with height function ϕ. Let $g : Y \to M$ be the isomorphism linking Ψ and Φ. We define the lower and upper local projection dimensions of the associated suspension flow for the observation f at a point $x \in M$:

$$\underline{d}_\mu^{f,g}(x) = \varliminf_{r \to 0} \frac{\log \mu(\pi(f \circ g)^{-1} B(f(x), r))}{\log r}$$

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and

$$\overline{d}_{\mu}^{f,g}(x) = \overline{\lim_{r \to 0}} \frac{\log \mu(\pi(f \circ g)^{-1} B(f(x), r))}{\log r},$$

where $\pi : Y \to X$ is the projection on X, i.e. for all $(u, t) \in Y$, $\pi(u, t) = u$.

Even if these dimensions are not identical to the local dimensions of the push-forward measure, in some cases we can link them (see §4 on the geodesic flow, for example). These dimensions appear naturally in our study. Indeed, when $f = \mathrm{id}$, i.e. when there is no observation, the return time of a point $(x, s) \in A \times B \subset Y$ in the set $A \times B$ under the action of the suspension flow is bounded from below by the return time of x in A under the action of T, and bounded from above by this same return time plus a constant (depending only on the height function ϕ). In [**6**], this idea was already used to prove that the asymptotic behaviour of the return time for a suspension flow of a point (x, s) in $B((x, s), r)$ was equal to the asymptotic behaviour of the return time of x in $B(x, r)$ under the action of T, which is linked to the asymptotic behaviour of the measure of $B(x, r)$. When there is an observation f, the return time of $f(x, s)$ in a set C corresponds to the return time of (x, s) in $f^{-1}C$ under the action of the suspension flow. So, it is natural to try to link the return time of (x, s) in $f^{-1}C$ to the measure of the projection on X of the set $f^{-1}C$. For more details, see Lemmas 7.2 and 7.3.

We recall the definition of the decay of correlations.

Definition 3.3. (X, T, μ) has a super-polynomial decay of correlations if, for all φ_1, φ_2 Lipschitz functions from X to $\mathbb{R}$ and for all $n \in \mathbb{N}^*$, we have

$$\left| \int_X \varphi_1 \circ T^n \varphi_2 \, d\mu - \int_X \varphi_1 \, d\mu \int_X \varphi_2 \, d\mu \right| \le \|\varphi_1\| \|\varphi_2\| \theta_n$$

with $\lim_{n\to\infty} n^k \theta_n = 0$ for all $k > 0$ and where $\| \cdot \|$ is the Lipschitz norm.

The second main theorem is the following.

THEOREM 3.4. *Let Ψ be a flow from M to M and ν an invariant probability measure for Ψ. Let f be a Lipschitz observation from M to $\mathbb{R}^N$. If Ψ is metrically isomorphic to a suspension flow Φ over T such that T has a super-polynomial decay of correlations and if the isomorphism g linking these two flows is Lipschitz, then*

$$\underline{R}_{\star}^{\Psi,f}(x) \ge \underline{d}_{\mu}^{f,g}(x) \quad \text{and} \quad \overline{R}_{\star}^{\Psi,f}(x) \ge \overline{d}_{\mu}^{f,g}(x)$$

for ν-almost every $x \in M$ such that $\underline{d}_{\mu}^{f,g}(x) > 0$.

Remark. We notice that the previous assumptions on the flow Ψ are satisfied when, for example, the flow Ψ is an Anosov flow and when the measure μ is an equilibrium state of a Hölder potential.

We recall the definition of an Anosov flow.

Definition 3.5. A differentiable flow $\Psi_t : M \to M$ is called an Anosov flow (or uniformly hyperbolic) if there exists a constant $0 < \lambda < 1$, a constant $C > 0$ and a decomposition of the tangent bundle

$$TM = E^u \oplus E^0 \oplus E^s \quad (\text{i.e. for all } x \in M,\ T_x M = E_x^s + E_x^0 + E_x^u),$$

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where E^0_x is generated by $d/dt(\Psi_t x)|_{t=0}$, E_u and E_s are sub-bundles $D\psi_t$-invariant for all $t \in \mathbb{R}$ (i.e. $D_x\Psi_t E^u_x = E^u_{\Psi_t x}$ and $D_x\Psi_t E^s_x = E^s_{\Psi_t x}$ for all $t \in \mathbb{R}$) such that for all $x \in M$ and for all $t > 0$,

$$\|d_x\Psi_t v\| \le C\lambda^t\|v\| \quad \text{for all } v \in E^s_x$$

and

$$\|d_x\Psi_{-t} v\| \le C\lambda^t\|v\| \quad \text{for all } v \in E^u_x.$$

When the observation is the identity map, we obtain an equality between recurrence rate and dimension (and for Anosov flows we obtain Theorem 1.3).

COROLLARY 3.6. *Let Ψ be a differentiable flow of M and ν an invariant probability measure for Ψ. If Ψ is metrically isomorphic to a suspension flow Φ over T such that T has a super-polynomial decay of correlations, and if the isomorphism g linking these two flows is Lipschitz, then*

$$\underline{R}^{\Psi}(x) = \underline{d}_\nu(x) - 1 \quad \textit{and} \quad \overline{R}^{\Psi}(x) = \overline{d}_\nu(x) - 1$$

for ν-almost every $x \in M$ not periodic and such that $\underline{d}_\nu(x) > 1$.

For C^1 flow with strictly positive entropy (for flows, the entropy corresponds to the entropy of the time-1 map, i.e. $h_\nu(\Psi) = h_\nu(\Psi_1)$), the lower local dimension satisfies another condition.

LEMMA 3.7. *Let Ψ be a flow C^1 on M and let ν be an invariant probability measure such that $h_\nu(\Psi) > 0$. Then, for ν-almost every $x \in M$,*

$$\underline{d}_\nu(x) > 1.$$

Proof. Let $\varepsilon > 0$. For every $x \in M$, we consider a submanifold N_x of dimension dim $M - 1$ and transverse to the flow Ψ. We now consider the family of balls $\{D_\delta(x)\}_{\varepsilon/2 \le \delta \le \varepsilon}$ of N_x, of centre x and diameter δ. Using the construction from [**6**, Lemma 5], we can prove that there exists $\delta_0 \in [\varepsilon/2, \varepsilon]$ such that $D_{\delta_0}(x)$ satisfies, for every $r > 0$ small enough,

$$\mu(\{y \in N_x : d(y, \partial D_{\delta_0}(x)) < r\}) \le cr, \tag{2}$$

where c is a constant depending only on ε and x and where μ is the induced measure by ν on $D_\varepsilon(x)$. We denote by $D(x)$ the set $D_{\delta_0}(x)$ and we consider the cylinders $C(x) = \bigcup_{0 \le t < \varepsilon} \Psi_t D(x)$. Since the cylinders $C(x)$ cover M, we can extract a finite subcovering $\bigcup_{i \in I} C_i$. Let $Z = \bigcup_{i \in I} D_i$, where D_i is the disc associated to the cylinder C_i for $i \in I$.

We define the transfer function $\varsigma : Z \to \mathbb{R}^+$ by

$$\varsigma(x) = \min\{t > 0 : \Psi_t x \in Z\}$$

and the transfer map $T : Z \to Z$ by

$$Tx = \Psi_{\varsigma(x)} x.$$

We remark that if $D_i \cap T^{-1} D_j \ne \emptyset$, then $T|_{D_i \cap T^{-1} D_j}$ is L-Lipschitz (Ψ_t being C^1 on M). Since μ is the invariant measure for T induced by ν, and since $h_\nu(\Psi) > 0$, we have $h_\mu(T) > 0$.

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Let ξ be a partition of Z finer than $\{D_i \cap T^{-1}D_j\}_{i,j\in I}$, of diameter arbitrarily small and satisfying, for every $0 < r < \operatorname{diam} \xi$,

$$\mu(x \in Z : d(x, \partial\xi) < r) \leq c_1 r, \tag{3}$$

where c_1 is a positive constant depending only on the diameter of ξ (the construction of the partition comes from [**6**] and (2)). Let us consider the set $A_n = \{x \in Z : d(T^n x, \partial\xi) < e^{-n}\}$. By (3), we have that for every n large enough, $\mu(A_n) \leq c_1 e^{-n}$. Then $\sum_{n\in\mathbb{N}} \mu(A_n) < +\infty$ and, by the Borel–Cantelli lemma, for μ-almost every x in Z, there exists $c(x) > 0$ such that for all $n \in \mathbb{N}$, $d(T^n x, \partial\xi) > c(x)e^{-n}$. This gives us that for μ-almost every $x \in Z$ and all $n \in \mathbb{N}$,

$$B_Z(x, c(x)L^{-n}e^{-n}) \subset \xi_n(x).$$

Finally, we observe that for every $x \in Z$, t and r small enough, denoting $y = \Psi_t x$, we have

$$B_M(y, r) \subset \bigcup_{|s|<r} \Psi_{t+s}(B_Z(x, L_1 r)),$$

where $L_1 = \sup_{0\leq s<r}\{\text{Lipschitz constant of } \Psi_{-s}\}$. Then, for μ-almost every x, for all t small enough and $y = \Psi_t x$, denoting $r_n = c(x)L_1^{-1}L^{-n}e^{-n}$ for n large enough, we have

$$\begin{aligned} \nu(B_M(y, r_n)) &\leq 2r_n \mu(B_Z(x, c(x)L^{-n}e^{-n})) \\ &\leq 2r_n \mu(\xi_n(x)). \end{aligned}$$

If the diameter of ξ is small enough, then $h_\mu(T, \xi) > (h_\mu(T)/2) > 0$, and for n large enough, $\mu(\xi_n) \leq e^{-nh_\mu(T)/2}$, and then

$$\underline{d}_\nu(y) \geq 1 + \frac{h_\mu(T)}{2(1 + \log L)} > 1.$$

Finally, choosing ε arbitrarily small, we obtain that for ν-almost every $y \in M$,

$$\underline{d}_\nu(y) > 1. \qquad \square$$

Remark. Using finer inequalities than in the previous proof, we can obtain that for ν-almost every $x \in M$,

$$\underline{d}_\nu(x) \geq 1 + h_\nu(\Psi)\bigg(\frac{1}{\Lambda_u(x)} - \frac{1}{\Lambda_s(x)}\bigg),$$

where $\Lambda_u(x)$ is the greatest Lyapunov exponent of the flow and where $\Lambda_s(x)$ the smallest Lyapunov exponent.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Bowen proved in [**10**] that an Anosov flow is metrically isomorphic to a hyperbolic symbolic flow (a suspension flow whose base is a subshift of finite type). We notice that the metric on X can be chosen such that the isomorphism linking these two flows is Lipschitz [**10**]. Since ν is an equilibrium state of a Hölder potential, μ is also an equilibrium state for the dynamical system $(X, \mathcal{A}, T)$, and then the decay of correlations is super-polynomial and the entropy is strictly positive. This allows us to use Corollary 3.6.

Moreover, for an Anosov flow, the number of periodic orbits is enumerable and since the measure is an equilibrium state, the set of periodic points has zero measure.

Finally, since the entropy is stricly positive, using Lemma 3.7, we have that for ν-almost every $x \in M$ $\underline{d}_\nu(x) > 1$, and Theorem 1.3 is proved. $\square$

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We emphasize that this result is an extension of the result for suspension flow in [**6**]. In fact, Theorem 3.4 is proved using the following theorem for suspension flow.

THEOREM 3.8. *Let* $(X, \mathcal{A}, \mu, T)$ *be a dynamical system with a super-polynomial decay of correlations and* Φ *the suspension flow over* T *with height function* ϕ. *Let* $\tilde{f} = f \circ g$ *be a Lipschitz function. Then we have*

$$\underline{R}_\star^{\Phi,\tilde{f}}(y) \geq \underline{d}_\mu^{f,g}(g(y)) \quad \textit{and} \quad \overline{R}_\star^{\Phi,\tilde{f}}(y) \geq \overline{d}_\mu^{f,g}(g(y))$$

for $\mu \otimes$ Leb-*almost every* $y \in Y$ *such that* $\underline{d}_\mu^{f,g}(g(y)) > 0$.

From now on, without loss of generality, let us assume that $\int_X \phi \, d\mu = 1$ and so the invariant measure for the suspension flow is $\nu_\mu = \mu \otimes$ Leb.

4. *Observations of the geodesic flow*

Let (M, h) be a C^∞ compact Riemannian manifold, i.e. M is a C^∞ compact manifold and h is a C^∞ Riemannian metric on M. The metric h induces a distance d on M. We suppose that the metric space (M, d) is complete. So, by the Hopf–Rinow theorem, M is geodesically complete. Let T^1M be the unit tangent bundle of (M, h). For $(p, v) \in T^1M$, $\gamma_{(p,v)} : \mathbb{R} \to M$ denotes the geodesic of initial conditions $\gamma_{(p,v)}(0) = p$ and $\dot{\gamma}_{(p,v)}(0) = v$. We now define the geodesic flow $\Psi_t : T^1M \to T^1M$ given by $\Psi_t(p, v) = (\gamma_{(p,v)}(t), \dot{\gamma}_{(p,v)}(t))$. Let Π be the canonical projection

$$\Pi : T^1M \longrightarrow M,$$
$$(p, v) \longmapsto p.$$

We recall that for every $(p, v) \in T^1M$, if $t > 0$ is small enough,

$$d(\Pi(p, v), \Pi(\Psi_t(p, v))) = t. \tag{4}$$

Let ν be an invariant probability measure for the geodesic flow.

THEOREM 4.1. *Let* Ψ *be the geodesic flow defined on* T^1M. *Then, for* ν*-almost every* $(p, v) \in T^1M$,

$$\underline{R}^\Psi(p, v) \leq \underline{d}_\nu(p, v) - 1 \quad \textit{and} \quad \overline{R}^\Psi(p, v) \leq \overline{d}_\nu(p, v) - 1.$$

Moreover, if M *has a strictly negative curvature and if* ν *is an equilibrium state of a Hölder potential, then*

$$\underline{R}^\Psi(p, v) = \underline{d}_\nu(p, v) - 1 \quad \textit{and} \quad \overline{R}^\Psi = \overline{d}_\nu(p, v) - 1$$

for ν*-almost every* $(p, v) \in T^1M$.

Proof. The first part of this theorem can be proved using the same technique as in Theorem 1.2 and using (4).

For the second part of the theorem, since M has a strictly negative curvature, Ψ is an Anosov flow and is metrically isomorphic to a hyperbolic symbolic flow [**10**]. Moreover, the metric on X can be chosen such that the isomorphism linking these two flows is Lipschitz. Since ν is an equilibrium state of a Hölder potential, μ is also an equilibrium state for the dynamical system $(X, \mathcal{A}, T)$, then the decay of correlations is super-polynomial. We can apply Corollary 3.6 and Lemma 3.7 (the entropy of the flow being strictly positive). □

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We are now interested in the return time of the position on the manifold M, i.e. the return time of the projection of the flow on the manifold.

THEOREM 4.2. *For ν-almost every $(p, v) \in T^1M$,*

$$\underline{R}^{\Psi,\Pi}(p, v) \leq \underline{d}_{\Pi_*\nu}(p) - 1 \quad \textit{and} \quad \overline{R}^{\Psi,\Pi}(p, v) \leq \overline{d}_{\Pi_*\nu}(p) - 1.$$

Remark. This result is still valid when the manifold M is not compact and when ν is a probability measure.

Proof of Theorem 4.2. The result comes from Theorem 2.4, using the escape function $\rho_r(p) = r$. In fact, it is important to note that by (4), for r small enough,

$$\inf\{t > 0 : \Pi(\Psi_t(p, v)) \notin B(p, r)\} = r. \qquad \square$$

Definition 4.3. A point $(p, v) \in T^1M$ is multiple if $t > 0$ exists such that $\Pi(\Psi_t(p, v)) = p$.

THEOREM 4.4. *If M has a strictly negative curvature and if ν is an equilibrium state of a Hölder potential, then*

$$\underline{R}^{\Psi,\Pi}(p, v) = \underline{d}_{\Pi_*\nu}(p) - 1 \quad \textit{and} \quad \overline{R}^{\Psi,\Pi}(p, v) = \overline{d}_{\Pi_*\nu}(p) - 1$$

for ν-almost every $(p, v) \in T^1M$ not multiple, such that $\underline{d}_{\Pi_*\nu}(p) > 1$.

Remark. For every point $p \in M$ there exist enumerably many $v \in T^1_pM$ such that (p, v) is multiple. We can wonder if, when the measure ν is an equilibrium state of a Hölder potential, the set of multiple points is a zero measure set.

Using a result of Ledrappier and Lindenstrauss [**15**], we obtain the following corollary.

COROLLARY 4.5. *If M is a compact Riemannian surface with striclty negative curvature, and if ν is an equilibrium state of a Hölder potential such that* $\dim_H \nu > 2$, *then*

$$R^{\Psi,\Pi}(p, v) = 1$$

for ν-almost every $(p, v) \in T^1M$ not multiple.

Proof. We just need to use Theorem 4.4 and [**15**, Theorem 1.1]. $\square$

Proof of Theorem 4.4. We already noticed in the proof of Theorem 4.1 that since M has a strictly negative curvature, Ψ is metrically isomorphic to a hyperbolic symbolic flow and that the metric on X can be choosen such that the isomorphism (denoted g) linking these two flows is Lipschitz. Moreover, since ν is an equilibrium state of a Hölder potential, μ is also an equilibrium state for the dynamical system $(X, \mathcal{A}, T)$ and the decay of correlations is super-polynomial. The end of the proof used the following lemma (which will be proved later).

LEMMA 4.6. *For every $(p, v) \in T^1M$,*

$$\underline{d}_\mu^{\Pi,g}(p, v) = \underline{d}_{\Pi_*\nu}(p) - 1$$

and

$$\overline{d}_\mu^{\Pi,g}(p, v) = \overline{d}_{\Pi_*\nu}(p) - 1.$$

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Since M is a compact manifold and since the flow is the geodesic flow on T^1M, $\beta > 0$ exists such that for every $t \leq \beta$ and for every $(x, u) \in T^1M$, $d(\Pi(x, u), \Pi(\Psi_t(x, u))) = t$. Then, for every r small enough, $\tau_r^{\Psi,\Pi}(p, v) \geq \beta$ and so for ν-almost every $(p, v) \in T^1M$ which is not multiple, $\tau_r^{\Psi,\Pi}(p, v) \to +\infty$ when $r \to 0$. Let $k \in \mathbb{N}^*$. For every point (p, v) not multiple there exists $r(k, p, v) > 0$ such that for every $0 < r < r(k, p, v)$, $\tau_r^{\Psi,\Pi}(p, v) > k$, which implies that $\tau_r^{\Psi,\Pi}(p, v) = \tau_{r,k}^{\Psi,\Pi,\star}(p, v)$. We obtain

$$\underline{R}^{\Psi,\Pi}(p, v) = \underline{R}_\star^{\Psi,\Pi}(p, v) \quad \text{and} \quad \overline{R}^{\Psi,\Pi}(p, v) = \overline{R}_\star^{\Psi,\Pi}(p, v). \tag{5}$$

The theorem is proved using (5), Theorems 2.4 and 3.8, and Lemma 4.6. □

We now prove the essential lemma used in the previous proof.

Proof of Lemma 4.6. Let $(p, v) \in T^1M$. Let $r > 0$.

$$\begin{aligned}\nu(\Pi^{-1}B(p, r)) &= \mu \otimes \mathrm{Leb}(g^{-1}(\Pi^{-1}B(p, r)))\\ &= \int_X \int_0^{\phi(x)} \mathbf{1}_{\Pi^{-1}B(p,r)}(g(x, t))\, dt\, d\mu(x).\end{aligned}$$

Since Ψ is the geodesic flow on T^1M, and since M is compact, $\beta > 0$ exists such that for every $t \leq \beta$, and for every $(p, u) \in T^1M$, $d(\Pi(p, u), \Pi(\Psi_t(p, u))) = t$. Let $x \in X$ be such that there exists $t \in (0, \phi(x))$ such that $g(x, t) \in \Pi^{-1}B(p, r)$. Then, if $r < \beta/2$, for every $s \in [2r, \beta]$, we have $g(x, t + s) \notin \Pi^{-1}B(p, r)$. Indeed,

$$g(x, t + s) = g(\Phi_s(x, t)) = \Psi_s(g(x, t))$$

and so

$$\begin{aligned}d(p, \Pi(g(x, t + s))) &\geq d(\Pi(g(x, t)), \Pi(g(x, t + s))) - d(p, \Pi(g(x, t)))\\ &\geq d(\Pi(g(x, t)), \Pi(\Psi_s g(x, t))) - r\\ &\geq s - r \geq 2r - r = r.\end{aligned}$$

Then

$$\nu(\Pi^{-1}B(p, r)) \leq \frac{\sup_x \phi(x)}{\beta} 2r\mu(\pi(\Pi \circ g)^{-1}B(p, r)). \tag{6}$$

Moreover, we notice

$$\begin{aligned}\nu(\Pi^{-1}B(p, 2r)) &= \int_X \int_0^{\phi(x)} \mathbf{1}_{\Pi^{-1}B(p,2r)}(g(x, t))\, dt\, d\mu(x)\\ &\geq \int_{\pi(\Pi\circ g)^{-1}B(p,r)} \int_0^{\phi(x)} \mathbf{1}_{\Pi^{-1}B(p,2r)}(g(x, t))\, dt\, d\mu(x).\end{aligned}$$

Let $x \in \pi(\Pi \circ g)^{-1}B(p, r)$, $t \in (0, \phi(x))$ exists such that $(x, t) \in (\Pi \circ g)^{-1}B(p, r)$. Then, if r is small enough,

$$\begin{aligned}d(\Pi(g(x, t + r)), p) &\leq d(\Pi(g(x, t + r)), \Pi(g(x, t))) + d(\Pi(g(x, t)), p)\\ &\leq d(\Pi(\Psi_r(g(x, t))), \Pi(g(x, t))) + r\\ &\leq r + r = 2r,\end{aligned}$$

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which gives

$$\nu(\Pi^{-1}B(p,2r)) \geq r\mu(\pi(\Pi\circ g)^{-1}B(p,r)). \tag{7}$$

Using (6) and (7), we obtain

$$\begin{aligned}\underline{d}_{\Pi_*\nu}(p) &= \varliminf_{r\to 0}\frac{\log \nu(\Pi^{-1}B(p,r))}{\log r}\\ &= \varliminf_{r\to 0}\frac{\log \mu(\pi(\Pi\circ g)^{-1}B(p,r))}{\log r}+1\\ &= \underline{d}_\mu^{\Pi,g}(p,v)+1\end{aligned}$$

and

$$\begin{aligned}\overline{d}_{\Pi_*\nu}(p) &= \varlimsup_{r\to 0}\frac{\log \nu(\Pi^{-1}B(p,r))}{\log r}\\ &= \varlimsup_{r\to 0}\frac{\log \mu(\pi(\Pi\circ g)^{-1}B(p,r))}{\log r}+1\\ &= \overline{d}_\mu^{\Pi,g}(p,v)+1.\end{aligned}$$

□

5. *Observations of suspension flows*

We now give some examples of observations of a suspension flow where we can compute the different dimensions and apply our theorems. In these examples, we will notice how important the choice of the escape function is.

Let $(X,\mathcal{A},\mu,d,T)$ be a metric measure-preserving system with a super-polynomial decay of correlations. We consider the suspension flow Φ over T with height function 1.

5.1. *Projection on the base.*

Let f be the observation of the projection on the base X, i.e.

$$\begin{aligned} f: Y &\longrightarrow X,\\ (x,s) &\longmapsto x.\end{aligned}$$

Then, for every $(x,s)\in Y$, we have

$$\underline{d}_{f_\star(\mu\otimes\text{Leb})}(f(x,s)) = \underline{d}_\mu(x) \quad\text{and}\quad \overline{d}_{f_\star(\mu\otimes\text{Leb})}(f(x,s)) = \overline{d}_\mu(x)$$

and

$$\underline{d}_\mu^{f,\text{id}}(x,s) = \underline{d}_\mu(x) \quad\text{and}\quad \overline{d}_\mu^{f,\text{id}}(x,s) = \underline{d}_\mu(x).$$

Choosing the escape function $\rho_r = |\log r|$, and comparing $\tau_{r,\rho}^{\Phi,f}$ to $\tau_{r,p}^{\Phi,f,\star}$, Theorems 2.4 and 3.4 give us that for $\mu\otimes$ Leb-almost every $(x,s)\in Y$,

$$\underline{R}_\rho^{\Phi,f}(x,s) = \underline{R}_\star^{\Phi,f}(x,s) = \underline{d}_\mu(x)$$

and

$$\overline{R}_\rho^{\Phi,f}(x,s) = \overline{R}_\star^{\Phi,f}(x,s) = \overline{d}_\mu(x)$$

for $\mu\otimes$ Leb-almost every $(x,s)\in Y$ such that $\underline{d}_\mu(x)>0$.

We emphasize that with this observation we are only interested in the return time of a point $x\in X$ under the action of T and that we obtain the results of [**23**] for discrete dynamical systems.

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We must note that, in this case, to obtain a non-trivial result the escape function must be chosen cautiously. Indeed, if the escape function is $\rho_r(x) = r$ for every x, for example, we obtain immediately that $\tau_{r,\rho}^{\Phi,f}(x, s) = r$ for every $(x, s) \in Y$ and for r small enough. This comes from the fact that $f^{-1}B(f(x, s), r) = B(x, r) \times [0, 1]$ and so the first escape time of $\Phi_t(x, s)$ of this set does not depend on r but only on (x, s). In this case, we cannot obtain an equality between recurrence rates and dimensions.

5.2. *Projection on time.* Let f now be the projection on time, i.e.

$$\begin{aligned} f : Y &\longrightarrow [0, 1), \\ (x, s) &\longmapsto s. \end{aligned}$$

Then, for every $(x, s) \in Y$, we have

$$\underline{d}_{f_\star(\mu\otimes\mathrm{Leb})}(f(x, s)) = 1 \quad \text{and} \quad \overline{d}_{f_\star(\mu\otimes\mathrm{Leb})}(f(x, s)) = 1.$$

Choosing the escape function $\rho_r(s) = 2r$ (which corresponds to, for example, the essential supremum of the escape times of $f^{-1}B(f(x, s), r)$), we obtain that for $\mu \otimes$ Leb-almost every $(x, s) \in Y$,

$$\underline{R}_\rho^{\Phi,f}(x, s) = \underline{R}_\star^{\Phi,f}(x, s) = \overline{R}_\rho^{\Phi,f}(x, s) = \overline{R}_\star^{\Phi,f}(x, s) = 0.$$

We can also see that in this case we can compute the projection dimension for the observation f:

$$\underline{d}_\mu^{f,\mathrm{id}}(x, s) = 0 \quad \text{and} \quad \overline{d}_\mu^{f,\mathrm{id}}(x, s) = 0.$$

This example and the previous example show us that for any flow and any observation, we have to choose an escape function depending on their parameters. Naturally, we can wonder if we can find a unique escape function giving optimal results for every flow. We notice that for the two previous examples, we obtain optimal results with the escape function

$$\rho_r(f(y)) = \frac{|\log r|}{f_*\nu(B(f(y), r))} \int_{f^{-1}B(f(y),r)} \inf\{t > 0 : f(\Phi_t(z)) \notin B(f(y), r)\}\, d\nu, \quad (8)$$

where $\nu = \mu \otimes \mathrm{Leb}$.

5.3. *Mixed projection.* Let us suppose that X is a product space, i.e. two spaces X_1 and X_2 exist such that $X = X_1 \times X_2$. Let f be the following observation:

$$\begin{aligned} f : Y &\longrightarrow X_1 \times [0, 1), \\ (x_1, x_2, s) &\longmapsto (x_1, s). \end{aligned}$$

Then, for every $(x_1, x_2, s) \in Y$, we have

$$\underline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s) = \varliminf_{r\to 0} \frac{\log \mu(B(x_1, r) \times X_2)}{\log r}$$

and

$$\overline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s) = \varlimsup_{r\to 0} \frac{\log \mu(B(x_1, r) \times X_2)}{\log r}.$$

Moreover, we can easily prove that

$$\underline{d}_{f_\star\mu\otimes\mathrm{Leb}}(f(x_1, x_2, s)) = \underline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s) + 1$$

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and

$$\overline{d}_{f_\star\mu\otimes\mathrm{Leb}}(f(x_1, x_2, s)) = \overline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s) + 1.$$

Then, with the escape function $\rho_r(x_1, s) = r|\log r|$ (which is in fact the escape function defined by (8)), and using Proposition 2.3 and Theorems 2.4 and 3.4, we can prove that for $\mu \otimes$ Leb-almost every $(x_1, x_2, s) \in Y$ such that $\underline{R}_\rho^{\Phi,f}(x_1, x_2, s) > 0$, we have

$$\underline{R}_\rho^{\Phi,f}(x_1, x_2, s) = \underline{R}_\star^{\Phi,f}(x_1, x_2, s) = \underline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s)$$

and

$$\overline{R}_\rho^{\Phi,f}(x_1, x_2, s) = \overline{R}_\star^{\Phi,f}(x_1, x_2, s) = \overline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s),$$

whenever $\underline{d}_\mu^{f,\mathrm{id}}(x_1, x_2, s) > 0$.

6. *Upper bound of the recurrence rates*

To prove Theorem 2.4, we first need the definition of a weakly diametrically regular measure.

Definition 6.1. A measure μ is weakly diametrically regular on the set $Z \subset X$ if for any $\eta > 1$, for μ-almost every $x \in Z$ and every $\varepsilon > 0$, $\delta > 0$ exists such that if $r < \delta$, then $\mu(B(x, \eta r)) \le \mu(B(x, r))r^{-\varepsilon}$.

Proof of Theorem 2.4. Since any probability measure is weakly diametrically regular on $\mathbb{R}^d$ (for any $d \in \mathbb{N}^*$) [**6**], the measure $f_*\nu$ is weakly diametrically regular on $\mathbb{R}^N$. We notice that in the definition of weakly diametrically regular measure and in Definition 2.1, the functions $\delta(f(\cdot), \varepsilon, \eta)$ and $\zeta(f(\cdot), \varepsilon, \xi_1, \xi_2)$ can be made measurable for every ε, η, ξ_1 and ξ_2. Let us fix $\varepsilon > 0$, $\eta = 4$, $\xi_1 = \frac{3}{4}$ and $\xi_2 = \frac{5}{4}$. We choose $\delta > 0$ small enough to have

$$\nu(M_\delta) > \nu(M) - \varepsilon = 1 - \varepsilon,$$

where

$$M_\delta := \{x \in M : \delta(f(x), \varepsilon, \eta) > \delta \text{ and } \zeta(f(x), \varepsilon, \xi_1, \xi_2) > \delta\}.$$

For $\delta > r > 0$, we define

$$A_\varepsilon(r) := \left\{ y \in M_\delta : \tau_{3r,\alpha}^{\Psi,f}(y)\nu(B(f(y), r))\frac{1}{\rho_{5r}(f(y))} \ge r^{-4\varepsilon} \right\}.$$

Let $C \subset M_\delta$ be such that $(f(x))_{x\in C}$ is a maximal r-separated set for $f(M_\delta)$. We have

$$\nu(A_\varepsilon(r)) \le \sum_{x\in C} \nu\left\{ y \in f^{-1}B(f(x), r) : \tau_{3r,\alpha}^{\Psi,f}(y)\nu(f^{-1}B(f(y), r))\frac{1}{\rho_{5r}(f(y))} \ge r^{-4\varepsilon} \right\}.$$

For $y \in f^{-1}B(f(x), 2r)$, we define

$$\tau_{2r,\alpha}^{\Psi,f}(y, x) = \inf\{t > \rho_{4r}(f(x))r^{-\varepsilon} : f(\Psi_t y) \in B(f(x), 2r)\}.$$

If $y \in f^{-1}B(f(x), r)$, since

$$\rho_{4r}(f(x))r^{-\varepsilon} \ge \rho_{3r}(f(y)),$$

and since

$$B(f(x), 2r) \subset B(f(y), 3r),$$

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we have

$$\tau^{\Psi,f}_{2r,\alpha}(y,x) \geq \tau^{\Psi,f}_{3r,\alpha}(y).$$

Moreover,

$$\nu(f^{-1}B(f(y),r)) \leq \nu(f^{-1}B(f(x),2r))$$

and

$$\rho_{4r}(f(x))r^{\varepsilon} \leq \rho_{5r}(f(y)),$$

which give us

$$\begin{aligned}\nu(A_\varepsilon(r)) & \\ &\leq \sum_{x\in C}\nu\bigg\{y\in f^{-1}B(f(x),r) : \tau^{\Psi,f}_{2r,\alpha}(y,x)\nu(f^{-1}B(f(x),2r))\frac{1}{\rho_{4r}(f(x))r^{\varepsilon}} \geq r^{-4\varepsilon}\bigg\}.\end{aligned}$$

Then, using Markov's inequality,

$$\nu(A_\varepsilon(r)) \leq \sum_{x\in C}\frac{r^{3\varepsilon}}{\rho_{4r}(f(x))}\nu(f^{-1}B(f(x),2r))\int_{f^{-1}B(f(x),2r)}\tau^{\Psi,f}_{2r,\alpha}(y,x)\,d\nu(y). \qquad (9)$$

We denote by D_r the set $f^{-1}B(f(x),2r)$. We define the application T_r from M to M by

$$\begin{aligned} T_r : M &\longrightarrow M,\\ y &\longmapsto \Psi_r(y)\end{aligned}$$

and the non-instantaneous return time for $y \in D_r$ by

$$\tau^{T_r}_{D_r}(y) := \inf\bigg\{n > \frac{\rho_{4r}(f(x))r^{-\varepsilon}}{r} : T_r^n y \in D_r\bigg\}.$$

Since this return time is inferior to the $(\rho_{4r}(f(x))r^{-\varepsilon}/r)$th return time of y in D_r, the Kač lemma gives

$$\int_{D_r}\tau^{T_r}_{D_r}(y)\,d\nu(y) \leq \frac{\rho_{4r}(f(x))r^{-\varepsilon}}{r}.$$

Moreover, for $y \in D_r$, we observe that

$$\tau^{\Psi,f}_{2r,\alpha}(y,x) \leq r\tau^{T_r}_{D_r}(y)$$

and so we obtain

$$\int_{f^{-1}B(f(x),2r)}\tau^{\Psi,f}_{2r,\alpha}(y,x)\,d\nu(y) \leq \rho_{4r}(f(x))r^{-\varepsilon}. \qquad (10)$$

Using (9) and (10), we have

$$\nu(A_\varepsilon(r)) \leq \sum_{x\in C} r^{2\varepsilon}\nu(f^{-1}B(f(x),2r)).$$

Since the measure $f_*\nu$ is weakly diametrically regular, and by definition of C, we obtain

$$\begin{aligned}\nu(A_\varepsilon(r)) &\leq \sum_{x\in C} r^{2\varepsilon}r^{-\varepsilon}\nu\bigg(f^{-1}B\bigg(f(x),\frac{r}{2}\bigg)\bigg)\\ &\leq r^{\varepsilon}.\end{aligned}$$

Finally, since

$$\sum_{n\in\mathbb{N}}\nu(A_\varepsilon(e^{-n})) < +\infty,$$

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by the Borel–Cantelli lemma, for ν-almost every $x \in M_\delta$, $N(x) \in \mathbb{N}$ exists such that for every $n > N(x)$,

$$\tau^{\Psi,f}_{3e^{-n},\alpha}(x)\nu(f^{-1}B(f(x), e^{-n})) \leq e^{4\varepsilon n}\rho_{5e^{-n}}(f(x))$$

and then

$$\frac{\log \tau^{\Psi,f}_{3e^{-n},\alpha}(x)}{n} \leq \frac{\log \nu(f^{-1}B(f(x), e^{-n}))}{-n} - \frac{\log \rho_{5e^{-n}}(f(x))}{-n} + 4\varepsilon. \tag{11}$$

Since we can easily prove that for $a > 0$,

$$\underline{R}^{\Psi,f}_\rho(x) = \liminf_{n\to+\infty} \frac{\log \tau^{\Psi,f}_{ae^{-n},\rho}(x)}{n}, \quad \overline{R}^{\Psi,f}_\rho(x) = \limsup_{n\to+\infty} \frac{\log \tau^{\Psi,f}_{ae^{-n},\rho}(x)}{n},$$

$$\begin{aligned}
&\underline{\lim}_{r\to 0} \frac{\log \nu(f^{-1}B(f(x), r))}{\log r} - \frac{\log \rho_r(f(x))}{\log r} \\
&\quad = \underline{\lim}_{n\to+\infty} \frac{\log \nu(f^{-1}B(f(x), e^{-n}))}{-n} - \frac{\log \rho_{5e^{-n}}(f(x))}{-n}, \\
&\overline{\lim}_{r\to 0} \frac{\log \nu(f^{-1}B(f(x), r))}{\log r} - \frac{\log \rho_r(f(x))}{\log r} \\
&\quad = \overline{\lim}_{n\to+\infty} \frac{\log \nu(f^{-1}B(f(x), e^{-n}))}{-n} - \frac{\log \rho_{5e^{-n}}(f(x))}{-n},
\end{aligned}$$

and since ε can be chosen arbitrarily small, the theorem is proved taking the limit superior or the limit inferior when $n \to +\infty$ in (11). □

7. *Lower bound of the recurrence rates*

In this section, we are going to prove Theorem 3.4, first proving Theorem 3.8.

Since the function $\tilde{f}$ is Lipschitz, we denote by $\tilde{L}$ its Lipschitz constant. Let $a > 0$, $b > 0$, $\beta > 0$ and $\eta > 0$. Let $Y_a := \{y \in Y, \underline{d}^{f,g}_\mu(g(y)) > a\}$. We define

$$\begin{aligned}
G_1 &= \{y \in Y_a : \text{for all } r \leq \eta,\ \mu(\pi \tilde{f}^{-1}B(\tilde{f}(y), r)) \leq r^a\}, \\
G_2 &= \left\{y = (x, s) \in Y_a : \text{for all } r \leq \eta,\ \mu\left(B\left(x, \frac{r}{2}\right)\right) \geq r^{N+b}\right\}, \\
G_3 &= \left\{y = (x, s) \in Y_a : \text{for all } r \leq \eta,\ \mu\left(B\left(x, \frac{r}{2}\right)\right) \geq \mu(B(x, 2r))r^\beta\right\}.
\end{aligned}$$

We notice that $G(a, b, \beta, \eta) := G_1 \cap G_2 \cap G_3$ satisfies

$$\mu \otimes \text{Leb}(G(a, b, \beta, \eta)) \underset{\eta\to 0}{\longrightarrow} \mu \otimes \text{Leb}(Y_a). \tag{12}$$

Indeed, by definition of $\underline{d}^{f,g}_\mu$, we have $\mu \otimes \text{Leb}(G_1) \to \mu \otimes \text{Leb}(Y_a)$. Moreover, since $\overline{d}_\mu \leq N$ μ-almost everywhere, $\mu \otimes \text{Leb}(G_2) \to \mu \otimes \text{Leb}(Y_a)$, and since the measure μ is weakly diametrically regular, $\mu \otimes \text{Leb}(G_3) \to \mu \otimes \text{Leb}(Y_a)$.

Let $\alpha > 0$; we define

$$Y(\alpha, a) = \{y = (x, s) \in Y_a : \alpha < s < \phi(x) - \alpha\}.$$

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We consider the set $G = G(a, b, \beta, \eta) \cap Y(\alpha, a)$. Following the ideas of Pène and Saussol [**19**] for return times in billiards, we will use a special cover of Y. Since X is compact, there exists, for small enough r, a finite subset $E = (m_i)_{i \in I} \subset X$ and a finite sequence $(s_{ij})_{(i,j) \in I \times J} \subset \mathbb{R}$ such that $\{P_{ij}(r) := \{\Phi_s(B(m_i, r) \times \{s_{ij}\}), 0 \leq s \leq r\}\}_{(i,j) \in I \times J}$ satisfies:
(1) $(m_i, s_{ij}) \in G$ for every $(i, j) \in I \times J$;
(2) $G \subset \bigcup_{(i,j) \in I \times J} P_{ij}(r)$;
(3) $B(m_{i_1}, r/2) \cap B(m_{i_2}, r/2) = \emptyset$ for every $i_1 \neq i_2$;
(4) $s_{ij} \in (0, \phi(m_i))$ for every i, j;
(5) $Y(\alpha, a) \subset \bigcup_{i,j} P_{ij}(r) \subset Y(\alpha/2, a)$; and
(6) $P_{ij}(r/2) \cap P_{kl}(r/2) = \emptyset$ for every $(i, j) \neq (k, l)$.
Let $r \leq \eta$; we define:

$$A_r^{\tilde{f}}(y) = \{x \in X : \exists t \in [0, \phi(x)[, \Phi_t(x, 0) \in \tilde{f}^{-1} B(\tilde{f}(y), r)\}.$$

LEMMA 7.1. *Under the hypothesis of Theorem 3.8, for every $(x, s) \in Y$, for every $n \in \mathbb{N}$, for every $K > 0$ and for every $r > 0$, we have*

$$\mu(B(x, r) \cap T^{-n} A_{Kr}^{\tilde{f}}(x, s)) \leq \frac{\tilde{L}c}{K} \frac{1}{r^2} \theta_n + \mu(B(x, 2r)) \mu(A_{2Kr}^{\tilde{f}}(x, s)),$$

where c is a strictly positive constant depending only on the different metrics.

Proof. Let $(x, s) \in Y$ and $r > 0$. Let $h_{x,r}$ and $h'_{(x,s),r}$ be the functions defined as follows:

$$\begin{aligned} h_{x,r} &: X \longrightarrow \mathbb{R}, \\ &u \longrightarrow h(u) = \max\left\{0, 1 - \frac{1}{r} d(u, B(x, r))\right\} \end{aligned}$$

and

$$\begin{aligned} h'_{(x,s),r} &: X \longrightarrow \mathbb{R}, \\ &u \longrightarrow \sup_{t \in [0, \phi(u)[} \max\left\{0, 1 - \frac{1}{Kr} d(B(\tilde{f}(x, s), Kr), \tilde{f}(u, t))\right\}, \end{aligned}$$

where $h_{x,r}$ is $1/r$-Lipschitz and $h'_{(x,s),r}$ is $\tilde{L}c/Kr$-Lipschitz with c the constant given by [**5**, Proposition 17]. Moreover, we have $1_{B(x,r)} \leq h_{x,r} \leq 1_{B(x,2r)}$ and $1_{A_{Kr}^{\tilde{f}}(x,s)} \leq h'_{(x,s),r} \leq 1_{A_{2Kr}^{\tilde{f}}(x,s)}$. Since the decay of correlations of (X, T, μ) is super-polynomial, we obtain

$$\begin{aligned} \mu(B(x, r) \cap T^{-n} A_{Kr}^{\tilde{f}}(x, s)) &\leq \int_X h_{x,r}(u) h'_{(x,s),r}(T^n u)\, d\mu(u) \\ &\leq \|h_{x,r}\| \|h'_{(x,s),r}\| \theta_n + \int_X h_{x,r}\, d\mu \int_X h'_{(x,s),r}\, d\mu \\ &\leq \frac{\tilde{L}c}{K} \frac{1}{r^2} \theta_n + \mu(B(x, 2r)) \mu(A_{2Kr}^{\tilde{f}}(x, s)). \end{aligned}$$

□

LEMMA 7.2. *Under the hypothesis of Theorem 3.8,*

$$\underline{R}_\star^{\Phi, \tilde{f}}(y) > 0 \text{ for } \mu \otimes \text{Leb-almost every } y \text{ such that } \underline{d}_\mu^{f,g}(g(y)) > 0.$$

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Proof. Let $Y_+ := \{\underline{d}_\mu^{f,g}(g(y)) > 0\}$. Let $1 > \varepsilon > 0$ and $a > 0$ be such that $\mu \otimes \mathrm{Leb}(Y_+) \geq \mu \otimes \mathrm{Leb}(Y_a) > \mu \otimes \mathrm{Leb}(Y_+) - \varepsilon$. Let $\alpha > 0$. We fix $b > 0$, $\beta = a/2$, and for $\eta > 0$ we consider the set $G = G(a, b, \beta, \eta) \cap Y(\alpha, a)$ defined previously. Let $n_0 \in \mathbb{N}$ be such that for all $n \geq n_0$, we have $\varepsilon_n = 1/n^{4/a} < \eta$ and we define

$$H_n := \{y = (x, s) \in Y(\alpha, a) : T^n x \in A_{r_n}^{\tilde{f}}(y)\}.$$

We consider the set $\{P_{ij}(r_n)\}_{(i,j)\in I\times J}$ defined previously. Then

$$\begin{aligned}\mu \otimes \mathrm{Leb}(G \cap H_n) &\leq \sum_{(i,j)\in I\times J} \mu \otimes \mathrm{Leb}((x, s) \in P_{ij}(r_n) : T^n x \in A_{r_n}^{\tilde{f}}(x, s))\\ &\leq \sum_{(i,j)\in I\times J} \mu \otimes \mathrm{Leb}((x, s) \in P_{ij}(r_n) : T^n x \in A_{(1+2\tilde{L}c)r_n}^{\tilde{f}}(m_i, s_{ij})).\end{aligned}$$

The definition of $P_{ij}(r_n)$ gives us for every $(i, j) \in I \times J$,

$$\begin{aligned}&\mu \otimes \mathrm{Leb}((x, s) \in P_{ij}(r_n) : T^n x \in A_{(1+2\tilde{L}c)r_n}^{\tilde{f}}(m_i, s_{ij}))\\ &\quad = \mu \otimes \mathrm{Leb}((x, s) \in B(m_i, r_n) \times [s_{ij}, s_{ij} + r_n] : T^n x \in A_{(1+2\tilde{L}c)r_n}^{\tilde{f}}(m_i, s_{ij}))\\ &\quad = r_n \mu(x \in B(m_i, r_n) : T^n x \in A_{(1+2\tilde{L}c)r_n}^{\tilde{f}}(m_i, s_{ij})). \qquad (13)\end{aligned}$$

Then, by Lemma 7.1 and (13), we obtain

$$\begin{aligned}&\mu \otimes \mathrm{Leb}(G \cap H_n)\\ &\quad \leq \sum_{(i,j)\in I\times J} r_n \left[\frac{\tilde{L}c}{2\tilde{L}c + 1}\frac{1}{r_n^2}\theta_n + \mu(B(m_i, 2r_n))\mu(A_{2(1+2\tilde{L}c)r_n}^{\tilde{f}}(m_i, s_{ij}))\right].\end{aligned}$$

By definition of G, we have

$$\begin{aligned}&\mu \otimes \mathrm{Leb}(G \cap H_n)\\ &\quad \leq \sum_{(i,j)\in I\times J} \left[\frac{\tilde{L}c}{2\tilde{L}c + 1} r_n^{-N-b-1}\theta_n + r_n^{1-\beta}(2(1 + 2\tilde{L}c)r_n)^a\right]\mu\left(B\left(m_i, \frac{r_n}{2}\right)\right)\end{aligned}$$

and by definition of P_{ij},

$$\mu \otimes \mathrm{Leb}(G \cap H_n) \leq \|\phi\| \left[\frac{\tilde{L}c}{2\tilde{L}c + 1} r_n^{-N-b-2}\theta_n + r_n^{-\beta}(2(1 + 2\tilde{L}c)r_n)^a\right].$$

Since $\sum_{n\in\mathbb{N}^*} r_n^{a-\beta} = \sum_{n\in\mathbb{N}^*}(1/n^2) < +\infty$ and since the decay of correlations is super-polynomial, we have

$$\sum_{n\in\mathbb{N}^*} \mu \otimes \mathrm{Leb}(G \cap H_n) < +\infty.$$

By the Borel–Cantelli lemma and using (12), we have for $\mu \otimes \mathrm{Leb}$-almost every $y = (x, s) \in Y(\alpha, a)$ that $n_1(y)$ exists such that for every $n \geq n_1(y)$, $T^n x \notin A_{1/n^{4/a}}^{\tilde{f}}$, i.e. for every $n \geq n_1(y)$ and for every $t \in [0, \phi(T^n x)[$, $\tilde{f}(T^n x, t) \notin B(\tilde{f}(x, s), 1/n^{4/a})$. Then, for $\mu \otimes \mathrm{Leb}$-almost every $y \in Y(\alpha, a)$, for every $p \geq n_1(y)$ and for every $n \geq n_1(y)$,

$$\tau_{1/n^{4/a},p}^{\Phi,\tilde{f},\star}(x, s) > n, \qquad (14)$$

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which gives

$$\underline{R}_\star^{\Phi,\tilde f}(y) = \lim_{p\to+\infty} \liminf_{r\to 0} \frac{\log \tau_{r,p}^{\Phi,\tilde f,\star}(y)}{-\log r}$$

$$= \lim_{p\to+\infty} \liminf_{n\to+\infty} \frac{\log \tau_{1/n^{4/a},p}^{\Phi,\tilde f,\star}(y)}{-\log 1/n^{4/a}}$$

$$\geq \lim_{p\to+\infty} \lim_{n\to+\infty} \frac{\log n}{\log n^{4/a}} = \frac{a}{4} > 0.$$

Since we can choose α and ε arbitrarily small, the lemma is proved. □

LEMMA 7.3. *Let* $a > 0$, $\delta > 0$ *and* $1 > \varepsilon > 0$. *For* $\mu \otimes \text{Leb}$*-almost every* $y \in Y_a := \{\underline{d}_\mu^{f,g}(g(y)) > a\}$, $r(y) > 0$ *exists such that for every* $r \in]0, r(y)[$ *and for every* $t \in [r^{-\delta}, \mu(\pi \tilde f^{-1} B(\tilde f(y), (2\tilde L c+1)er))^{-1+\varepsilon}]$, *we have* $d(\tilde f(\Phi_t y), \tilde f(y)) \geq r$.

Proof. Let $\alpha > 0$. Let $1 > \varepsilon > 0$. We fix $b > 0$, $\beta = a\varepsilon/2$ and for $\eta > 0$ we consider the set $G = G(a, b, \beta, \eta) \cap Y(\alpha, a)$. Let $\delta > 0$ and $r \leq \eta$; we define

$$C_\varepsilon(r) := \{w \in Y : \exists t \in [r^{-\delta}, \mu(\pi \tilde f^{-1} B(\tilde f(w), (6\tilde L c + 2)r))^{-1+\varepsilon}] \text{ such that } d(\tilde f(\Phi_t w), \tilde f(w)) < r\}.$$

For $x \in G$ and $s > 0$, we define $B_{r,s}(x) := B(x, r) \times [s, s + r]$ and, defining

$$I_r = [\lfloor r^{-\delta} \rfloor, \mu(\pi \tilde f^{-1} B(\tilde f(x, s), (4\tilde L c + 2)r))^{-1+\varepsilon}] \cap \mathbb{N},$$

we can prove that we have

$$B_{r,s}(x) \cap C_\varepsilon(r) \subset \bigcup_{I_r} B(x, r) \cap T^{-n} A^{\tilde f}_{(2\tilde L c+1)r}(x, s) \times [s, s + r]. \tag{15}$$

Indeed, let us choose an element w of $B_{r,s}(x) \cap C_\varepsilon(r)$. By definitition, this means that $w \in B_{r,s}(x)$ and that there exists $t \in [r^{-\delta}, \mu(\{\pi \tilde f^{-1} B(\tilde f(w), (6\tilde L c + 2)r)\})^{-1+\varepsilon}]$ such that $d(\tilde f(\Phi_t w), \tilde f(w)) < r$.

However, noticing that $B(\tilde f(x, s), (4\tilde L c + 2)r) \subset B(\tilde f(w), (6\tilde L c + 2)r)$, the previous time t is smaller than $\mu(\{\pi \tilde f^{-1} B(\tilde f(x, s), (4\tilde L c + 2)r)\})^{-1+\varepsilon}$.

Now, since we work with a suspension flow over an application T, we denote $w = (v, u)$ and we can divide our interval in which t exists in time intervals of length 1 and with integer bounds.

Then, if $(v, u) \in B_{r,s}(x)$, then an integer $n \in I_r$ exists and there exists a time $t \in [0, \phi(T^n v)[$ such that we have $d(\tilde f(T^n v, t), \tilde f(v, u)) < r$.

Noticing that $\tilde f$ is Lipschitz and that $(v, u) \in B_{r,s}(x)$, we can see that for the previous n and t, we also have $d(\tilde f(T^n v, t), \tilde f(x, s)) < (2\tilde L c + 1)r$. This implies that $v \in T^{-n} A^{\tilde f}_{(2\tilde L c+1)r}(x, s)$ and so we obtain equation (15).

Let $k > 1$ be such that $\delta(k - 1) - 1 \geq N + 2b$ and let $\eta > 0$ be such that $n \geq \eta^{-\delta}$ implies $(k - 1)(n + 1)^{-k} \geq \theta_n$ (which is possible by definition of θ_n). We have,

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by Lemma 7.1,

$$\begin{aligned}\mu \otimes \mathrm{Leb}(B_{r,s}(x) \cap C_\varepsilon(r)) &\\ &\le r \sum_{n \in I_r} \bigg[\frac{\tilde{L}c}{(2\tilde{L}c+1)r^2} \theta_n + \mu(B(x,2r))\mu(A^{\tilde{f}}_{2(2\tilde{L}c+1)r}(x,s)) \bigg] \\ &\le \frac{\tilde{L}c}{(2\tilde{L}c+1)} r^{\delta(k-1)-1} + r\mu(B(x,2r))\mu(A^{\tilde{f}}_{2(2\tilde{L}c+1)r}(x,s))^{\varepsilon}\end{aligned}$$

and, by definition of G,

$$\begin{aligned}\mu \otimes \mathrm{Leb}(B_{r,s}(x) \cap C_\varepsilon(r)) &\le \frac{\tilde{L}c}{(2\tilde{L}c+1)} r^{N+2b} + (4\tilde{L}c+2)^{a\varepsilon} r^{1+(a\varepsilon/2)} \mu\bigg(B\bigg(x, \frac{r}{2}\bigg)\bigg) \\ &\le \mu\bigg(B\bigg(x, \frac{r}{2}\bigg)\bigg)\bigg(\frac{\tilde{L}c}{2\tilde{L}c+1} r^b + (4\tilde{L}c+2)^{a\varepsilon} r^{1+(a\varepsilon/2)}\bigg).\end{aligned}$$

We consider the set $\{P_{ij}(r)\}_{(i,j)\in I\times J}$ defined previously. We have

$$\begin{aligned}\mu \otimes \mathrm{Leb}(G \cap C_\varepsilon(r)) &\le \sum_{(i,j)\in I\times J} \mu \otimes \mathrm{Leb}(C_\varepsilon(r) \cap P_{ij}(r)) \\ &\le \sum_{(i,j)\in I\times J} \mu\bigg(B\bigg(m_i, \frac{r}{2}\bigg)\bigg)\bigg(\frac{\tilde{L}c}{2\tilde{L}c+1} r^b + (4\tilde{L}c+2)^{a\varepsilon} r^{1+(a\varepsilon/2)}\bigg) \\ &\le \sum_{j\in J} \bigg(\frac{\tilde{L}c}{2\tilde{L}c+1} r^b + (4\tilde{L}c+2)^{a\varepsilon} r^{1+(a\varepsilon/2)}\bigg) \\ &\le \frac{\|\phi\|}{r} \bigg(\frac{\tilde{L}c}{2\tilde{L}c+1} r^b + (4\tilde{L}c+2)^{a\varepsilon} r^{1+(a\varepsilon/2)}\bigg) \\ &\le \frac{\tilde{L}c}{2\tilde{L}c+1} r^{b-1} + (4\tilde{L}c+2)^{a\varepsilon} r^{(a\varepsilon/2)}.\end{aligned}$$

Then, choosing $b = 1$, we obtain

$$\sum_{k\in\mathbb{N}} \mu \otimes \mathrm{Leb}(G \cap C_\varepsilon(e^{-k})) < +\infty.$$

So, by the Borel–Cantelli lemma, for $\mu \otimes \mathrm{Leb}$-almost every $y \in G$, $n_1(y)$ exists such that for every $k \ge n_1(y)$, $y \notin C_\varepsilon(e^{-k})$. Then, for r small enough, $k \in \mathbb{N}$ exists such that $e^{-k-1} < r \le e^{-k} \le e^{-n_1(y)}$. Moreover, since $e^{\delta k} \le r^{-\delta}$ and $(2\tilde{L}c+1)e^{-k} < (2\tilde{L}c+1)er$, an integer $t \in [r^{-\delta}, \mu(\pi \tilde{f}^{-1} B(\tilde{f}(y), (2\tilde{L}c+1)er))^{-1+\varepsilon}]$ does not exist such that $d(\tilde{f}(\Phi_t y), \tilde{f}(y)) \ge r$. The lemma is proved, choosing α arbitrarily small. □

Proof of Theorem 3.8. Let $\zeta > 0$. Since $\underline{R}^{\Phi,\tilde{f}}_\star(y) > 0$ for $\mu \otimes \mathrm{Leb}$-almost every $y \in Y_+ = \{\underline{d}^{f,g}_\mu(g(y)) > 0\}$, by Lemma 7.2, $a > 0$ exists such that

$$\mu \otimes \mathrm{Leb}(Y_+) \ge \mu \otimes \mathrm{Leb}(\{\underline{R}^{\Phi,\tilde{f}}_\star(y) > a\}) > \mu \otimes \mathrm{Leb}(Y_+) - \zeta.$$

For every $y \in \{\underline{R}^{\Phi,\tilde{f}}_\star(y) > a\}$, for p large enough and for r small enough, we have

$$\tau^{\Phi,\tilde{f},\star}_{r,p}(y) \ge r^{-a}.$$

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Thanks to Lemma 7.3, choosing $\delta = a$ and $\varepsilon > 0$, for $\mu \otimes \text{Leb}$-almost every $y \in \{\underline{R}_\star^{\Phi,\tilde{f}}(y) > a\}$, if r is small enough and p is large enough, then

$$\tau_{r,p}^{\Phi,\tilde{f},\star}(y) \geq \mu(\pi \tilde{f}^{-1} B(\tilde{f}(y), (2\tilde{L}c + 1)er))^{-1+\varepsilon}.$$

Then $\mu \otimes \text{Leb}$-almost everywhere on $\{\underline{R}_\star^{\Phi,\tilde{f}}(y) > a\}$, we have $\underline{R}_\star^{\Phi,\tilde{f}}(y) \geq (1 - \varepsilon)\underline{d}_\mu^{f,g}(g(y))$ and $\overline{R}_\star^{\Phi,\tilde{f}}(y) \geq (1 - \varepsilon)\overline{d}_\mu^{f,g}(g(y))$. The theorem is proved choosing $\varepsilon > 0$ arbitrarily small and then $\zeta > 0$ arbitrarily small. □

Proof of Theorem 3.4. There exists $Y_g \subset Y$ such that $\mu \otimes \text{Leb}(Y_g) = \nu(g(Y_g)) = 1$ and such that g is one-to-one on Y_g. Then, for ν-almost every $x \in M$, there exists $y \in Y_g$ such that $x = g(y)$, and noticing that

$$\underline{R}_\star^{\Psi,f}(x) = \underline{R}_\star^{\Phi,\tilde{f}}(y)$$

and that

$$\overline{R}_\star^{\Psi,f}(x) = \overline{R}_\star^{\Phi,\tilde{f}}(y),$$

the theorem is proved using Theorem 3.8. □

Proof of Corollary 3.6. Let $x \in M$ be a non-fixed point. We have already seen in the proof of Theorem 1.2 that by the flow box theorem, there exists a neighbourhood U of x, $\beta > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ such that for every $0 < t \leq \beta$ and for every $z \in U$, $\gamma_2 t \geq d(z, \Psi_t(z)) \geq \gamma_1 t$. Let $\beta/2 > r > 0$ be such that $B(x, r) \subset U$.

$$\begin{aligned}\nu(B(x, r)) &= \mu \otimes \text{Leb}(g^{-1}B(x, r))\\ &= \int_X \int_0^{\phi(u)} \mathbf{1}_{B(x,r)}(g(u, t))\, dt\, d\mu(u).\end{aligned}$$

Let $u \in X$ be such that there exists $t \in (0, \phi(u))$ satisfying $g(u, t) \in B(x, r)$. Then, since $r < \beta/2$, for every $s \in [2r/\gamma_1, \beta]$, we have $g(u, t + s) \notin B(x, r)$. Indeed,

$$g(u, t + s) = g(\Phi_s(u, t)) = \Psi_s(g(u, t))$$

and so

$$\begin{aligned}d(x, g(u, t + s)) &\geq d(g(u, t), g(u, t + s)) - d(x, g(u, t))\\ &\geq d(g(u, t), \Psi_s g(u, t)) - r\\ &\geq \gamma_1 s - r \geq 2r - r = r.\end{aligned}$$

Then

$$\nu(B(x, r)) \leq 2r \frac{\sup_u \phi(u)}{\gamma_1} \mu(\pi g^{-1} B(x, r)). \tag{16}$$

Moreover, we observe that

$$\begin{aligned}\nu(B(x, 2r)) &= \int_X \int_0^{\phi(u)} \mathbf{1}_{B(x,2r)}(g(u, t))\, dt\, d\mu(u)\\ &\geq \int_{\pi g^{-1}B(x,r)} \int_0^{\phi(u)} \mathbf{1}_{B(x,2r)}(g(u, t))\, dt\, d\mu(u).\end{aligned}$$

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Let $u \in \pi g^{-1}B(x,r)$; $t \in (0, \phi(u))$ exists such that $(u,t) \in g^{-1}B(x,r)$. Then, if r is small enough,

$$\begin{aligned} d\left(g\left(u, t+\frac{r}{\gamma_2}\right), x\right) &\leq d\left(g\left(u, t+\frac{r}{\gamma_2}\right), g(u,t)\right) + d(g(u,t),x) \\ &\leq d(\Psi_{r/\gamma_2}(g(u,t)), g(u,t)) + r \\ &\leq r + r = 2r, \end{aligned}$$

which gives

$$\nu(B(x,2r)) \geq \frac{r}{\gamma_2}\mu(\pi g^{-1}B(x,r)). \tag{17}$$

Using (16) and (17), we obtain

$$\underline{d}_\nu(x) = \varliminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = \varliminf_{r\to 0} \frac{\log \mu(\pi g^{-1}B(x,r))}{\log r} + 1 = \underline{d}_\mu^{\mathrm{id},g}(x) + 1 \tag{18}$$

and

$$\overline{d}_\nu(x) = \varlimsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = \varlimsup_{r\to 0} \frac{\log \mu(\pi g^{-1}B(x,r))}{\log r} + 1 = \overline{d}_\mu^{\mathrm{id},g}(x) + 1. \tag{19}$$

Since for every $t \leq \beta$ and for every $y \in M$, $d(y, \Psi_t(y)) \geq \gamma t$, then, for every r small enough, $\tau_r^\Psi(x) \geq \gamma\beta$. So, for ν-almost every $x \in M$ which is not periodic, $\tau_r^\Psi(x) \to +\infty$ when $r \to 0$. Let $p \in \mathbb{N}^*$. Then, for every non-periodic point x, $r(p,x) > 0$ exists such that for every $0 < r < r(p,x)$, $\tau_r^\Psi(x) > p$, which implies that $\tau_r^\Psi(x) = \tau_{r,p}^{\Psi,\mathrm{id},\star}(x)$. We obtain

$$\underline{R}^\Psi(x) = \underline{R}_\star^{\Psi,\mathrm{id}}(x) \quad \text{and} \quad \overline{R}^\Psi(x) = \overline{R}_\star^{\Psi,\mathrm{id}}(x). \tag{20}$$

Finally, the corollary is proved by (18), (19), (20), and Theorems 1.2 and 3.4. □

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