EXPONENTIAL LAW FOR RANDOM MAPS ON COMPACT MANIFOLDS

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Abstract. We consider random dynamical systems on manifolds modeled by a skew product which have certain geometric properties and whose measures satisfy quenched decay of correlations at a sufficient rate. We prove that the limiting distribution for the hitting and return times to geometric balls are both exponential for almost every realisation. We then apply this result to random C^2 maps of the interval, random parabolic maps on the unit interval and random perturbation of partially hyperbolic attractors on a compact Riemannian manifold.

1. Introduction

As a generalisation of deterministic dynamical systems but also as a better approximation of natural phenomena (e.g. existence of smalls perturbations), random dynamical systems have been extensively studied in the last few decades. Unlike deterministic systems which only consider the iteration of one map, random systems allow the composition of different maps (for example by adding random noise or random perturbations) thus increasing the difficulty of analyzing their statistical properties, in particular since these maps generally do not share a common invariant measure. We point the readers to the review paper by Kifer and Liu [20] for more details.

Among these statistical properties, we would like to focus on limit laws for rare events, more precisely Hitting Time Statistics (HTS) and Return Time Statistics (RTS). The study of hitting/return times for deterministic systems traces all the way back to the famous work of Poincaré [22] who proved that, in a finite measure preserving dynamical system, the orbit of almost every point comes back as close as you want to its starting point. Our main subject of interest will be the time needed for the orbit to come back. More precisely, if we denote by $\tau_A(x)$ the first time that the orbit of x enters the set A,

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then one can consider the functions

$$F_A^h(t) = \mu\left(x \in X : \tau_A(x) > \frac{t}{\mu(A)}\right)$$

and

$$F_A^r(t) = \frac{1}{\mu(A)}\mu\left(x \in A : \tau_A(x) > \frac{t}{\mu(A)}\right)$$

where μ is an invariant measure of the transformation. We can observe that the scaling factor $\frac{1}{\mu(A)}$ is suggested by Kac's Lemma [19] which says that $\int_A \tau_A \, d\mu = 1$.

One is naturally interested whether the functions $F_{A_n}^h$ (respectively $F_{A_n}^r$) converge to a limiting function F^h (respectively F^r) as $n \to \infty$ when one chooses a sequence of nested sets $\{A_n\}_{n=1}^{\infty}$. Indeed, if the sets A_n are taken to be cylinder sets with respect to a generating measurable partition, then the limit is known to be exponential (i.e. $F^h(t) = F^r(t) = e^{-t}$ for all $t > 0$) for non-periodic points for mixing measures (see e.g. [12, 1]). In the case of Bowen balls the same result is known to be true [16]. For geometric balls $B_r(y)$ it has been proven that the limit F is exponential if μ has exponential decay of correlations (e.g. [23] and references therein). We refer to the reviews [13, 26] for more details on this subject.

Our goal in this paper is to extend some of these results for deterministic dynamical systems to the realm of random dynamical systems. We consider a family of maps $\{T_\omega\}_\omega$ with $T_\omega : X \to X$. The randomness comes from a dynamical system (Ω, θ, ν) and the random orbit is given by

$$T_\omega^n(x) = T_{\theta^n\omega} \circ T_{\theta^{n-1}\omega} \circ \cdots \circ T_\omega(x).$$

Thus, one can define the (quenched) hitting time $\tau_A^\omega(x)$ as the first time the random orbit of x enters the set A. There are two ways to define the hitting times distribution, namely

$$F(t) = \mathbb{P}\left((\omega, x) : \tau_A^\omega(x) > \frac{t}{\mu(A)}\right)$$

and

$$F^\omega(t) = \mu^\omega\left(x : \tau_A^\omega(x) > \frac{t}{\mu(A)}\right).$$

The first is known as the annealed distribution, where the probability is taken with respect to the measure $\mathbb{P}$, which is invariant for the random dynamical system, i.e. invariant for the associated skew product. The second is called the quenched distribution where the probability is taken with the measure μ^ω associated with the 'realisation' ω. In both cases, the scaling factor is $\frac{1}{\mu(A)}$, where μ is the marginal measure and is suggested by Kac's Lemma for the associated skew product [21, 24].

In [5, 3, 23], it is proven that the annealed distribution for geometric balls converges to an exponential for maps with (annealed) exponential decay of correlations. In the first two papers, their method exploits the relation between the hitting times statistics and the extreme value distribution, while in the third one, following [18, 26], the error between the distribution of the hitting times and the exponential law is controlled using recurrence rates. In these papers, the convergence of the annealed return time distribution to an exponential is also proven.

On the other hand, a quenched result is more interesting since it easily implies the annealed result by integrating over ω, but more difficult to get. The only known results are [24, 25, 11] where random subshifts of finite type with fast decay of correlations are considered. More precisely, in [24, Theorem 1] an exponential law is proved for hitting times in this setting. Here, we extend this result to maps and prove that the quenched hitting time statistics and the quenched return time statistics converge, for almost every ω, to the exponential distribution for random maps which have certain geometric properties and with some rapidly mixing conditions.

We emphasize that this is the first paper where one manages to obtain the convergence of the quenched return time statistics to the exponential distribution. Indeed, in the articles studying the quenched case, they did not prove the distribution for the return times. More importantly, one can notice that the convergence of the return times distribution does not come immediately from the convergence of the hitting times distribution, as one could have hoped for from the deterministic case (e.g. [14, 26]) or the annealed case [23]. In these works, the main idea was to observe that the error between the distribution of the hitting times and the distribution of the return times was controlled by the error between the distribution of the hitting times and the exponential law. Thus, obtaining the convergence of the HTS implied immediately the convergence of the RTS. In the quenched case, since the sample measures are not invariant, even if the error terms seem of similar order, they are of different nature as one will see in equations (6) and (8). Indeed, the difference between the distribution of the hitting times and the exponential law is controlled by an error term which is a finite Birkhoff sum of elements (depending on the iterates of ω) while the difference between the distribution of the hitting times and the distribution of the return times is controlled by an error term which only depends on ω.

The main theorems are stated in Section 2 and proven in Sections 3 to 6. The proof is based on the deterministic case [17] which in its turn was derived from the deterministic case on Young towers [15]. We emphasize that our proof is not just a simple adaptation of the deterministic case to the random case. In particular, one needs to be especially careful in the study of the very short returns (Section 3) and we encourage the reader to pay attention to Proposition 3.1 which is a complex adaptation of a lemma of [10] on the measure of the set of very short returns. One of the main difficulties in the random setting is the observation that T_ω^{2k} is generally different from $T_\omega^k \circ T_\omega^k$. Consequently, a return to $B_r(x)$ at time k under the random map T_ω^k does not imply the existence of a return to $B_{Cr}(x)$ at time $2k$, even for C sufficiently large.[5]

In Section 7 we consider three examples, namely random C^2 expanding interval maps and random Pomeau-Manneville maps where we use the derivation of the fibered measures from [6] and the decay of sequential systems for parabolic maps [2]; the final example is the random perturbation of partially hyperbolic attractors, where the system exhibits non-trivial stable and unstable leaves depending on ω.

[5]However, when the system is deterministic, $B_r(x) \cap T^k B_r(x) \neq \emptyset$ shows that $T^k B_r(x) \cap T^{2k} B_r(x) \neq \emptyset$ and consequently $B_{Cr}(x) \cap T^{2k} B_{Cr}(x)$ for C large enough. This is the key argument behind Lemma B.3 of [10].

2. Random Maps

Let $\theta : \Omega \to \Omega$ be the two-sided shift map on a full shift space Ω with θ-invariant ergodic probability measure ν. Let M be a compact manifold and for every $\omega \in \Omega$, let $T_\omega : M \to M$ be a measurable map where we assume that $T_\omega = T_{\omega_0}$.

The skew product S on $\Omega \times M$ is then given by $S(\omega, x) = (\theta\omega, T_\omega x)$. For the iterates we obtain $S^n(\omega, x) = (\theta^n\omega, T^n_\omega x)$ where $T^n_\omega = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$.

Assume that $\mathbb{P}$ is a measure on $\Omega \times M$ invariant under the skew action S and with marginal ν on Ω. There is a class of measures μ^ω for $\omega \in \Omega$ on M, such that $d\mathbb{P} = d\mu^\omega d\nu(\omega)$. These measures satisfy the equivariance property $T^*_\omega \mu^\omega = \mu^{\theta\omega}$ for ν-almost every $\omega \in \Omega$. We denote by $\mu = \int_\Omega \mu^\omega \, d\nu(\omega)$ the marginal measure on M.

For every realisation $\omega \in \Omega$, let $\Gamma^u(\omega)$ be a collection of unstable leaves $\gamma^u(\omega)$ and $\Gamma^s(\omega)$ a collection of stable leaves $\gamma^s(\omega)$. We assume that $\gamma^u \cap \gamma^s$ consists of a single point for all $(\gamma^u, \gamma^s) \in \Gamma^u \times \Gamma^s$. The map T_ω contracts along the stable leaves and similarly the inverse branches of T_ω contract along the unstable leaves.

For an unstable leaf $\gamma^u(\omega)$ denote by $\mu^\omega_{\gamma^u}$ the disintegration of μ^ω with respect to the γ^u. We assume that μ^ω has a product like decomposition $d\mu^\omega = d\mu^\omega_{\gamma^u} d\upsilon^\omega(\gamma^u)$, where υ^ω is a transversal measure. That is, if f is a function on M then

$$\int f(x)\, d\mu^\omega(x) = \int_{\Gamma^u(\omega)} \int_{\gamma^u} f(x)\, d\mu^\omega_{\gamma^u}(x)\, d\upsilon^\omega(\gamma^u).$$

If $\gamma^u, \hat\gamma^u \in \Gamma^u(\omega)$ are two unstable leaves then the holonomy map $\mathcal{H} : \gamma^u \to \hat\gamma^u$ is defined such that by $\mathcal{H}(x)$ is the unique point of intersection between $\hat\gamma^u$ and $\gamma^s(x)$ for $x \in \gamma^u$, where $\gamma^s(x)$ is the local stable leaf through x.

Let us denote by $J^\omega_n = \frac{dT^n_\omega \mu^\omega_{\gamma^u}}{d\mu^\omega_{\gamma^u}}$ the Jacobian of the map T^n_ω with respect to the measure μ^ω in the unstable direction.

Fix ω and let γ^u be a local unstable leaf. Assume there exist $R > 0$ and for every $n \in \mathbb{N}$ finitely many $y_k \in T^n_\omega \gamma^u$ so that $T^n_\omega \gamma^u \subset \bigcup_k B_{R,\gamma^u}(y_k)$, where $B_{R,\gamma^u}(y)$ is the embedded R-disk centered at y in the unstable leaf γ^u. Denote by $\zeta_{\varphi,k} = \varphi(B_{R,\gamma^u}(y_k))$ where $\varphi \in \mathscr{I}^\omega_n$ and $\mathscr{I}^\omega_n$ denotes the inverse branches of T^n_ω. We call ζ an n-cylinder. Then there exists a constant L so that the number of overlaps $N_{\varphi,k} = |\{\zeta_{\varphi',k'} : \zeta_{\varphi,k} \cap \zeta_{\varphi',k'} \neq \varnothing, \varphi' \in \mathscr{I}^\omega_n\}|$ is bounded by L for all $\varphi \in \mathscr{I}^\omega_n$ and for all k and n. This follows from the fact that $N_{\varphi,k}$ equals $|\{k' : B_{R,\gamma^u}(y_k) \cap B_{R,\gamma^u}(y_{k'}) \neq \varnothing\}|$ which is uniformly bounded by some constant L.

To obtain an exponential law for the distribution of hitting time and return time, we need a few assumptions. First of all, we need information on the annealed and quenched decay of correlations:

(I) There exists a decay function $\lambda(k)$ so that

$$\left| \int_\Omega \int_M G(H \circ T^k_\omega)\, d\mu^\omega\, d\nu(\omega) - \mu(G)\mu(H) \right| \le \lambda(k) \|G\|_{Lip} \|H\|_\infty \qquad \forall k \in \mathbb{N}$$

for every $G \in Lip(M, \mathbb{R})$ and $H \in L^\infty(M, \mathbb{R})$.

(II) For ν-almost every ω, the individual measure μ^ω has the following decay of correlations

$$\left| \int_M G(H \circ T^k_\omega)\, d\mu^\omega - \mu^\omega(G)\mu^{\theta^k\omega}(H) \right| \le \lambda(k) \|G\|_{Lip} \|H\|_\infty \qquad \forall k \in \mathbb{N},$$

for every $H \in L^\infty(M,\mathbb{R})$ which are constant on local stable leaves γ^s of T_ω and for every $G \in Lip(M,\mathbb{R})$.
Then, we need some geometric assumptions:
(III) (Distortion) For ν-almost every ω, we require that $\frac{J_n^\omega(x)}{J_n^\omega(y)} = \mathcal{O}(\Theta(n))$ for all $x, y \in \zeta$ and n, where ζ are n-cylinders in unstable leaves γ^u and Θ is a non-decreasing function which below we assume to be $\Theta(n) = \mathcal{O}(n^{\kappa'})$ for some $\kappa' \geq 0$.
(IV) (Contraction) There exists a function $\delta(n) \to 0$ which decays at least summably polynomially, i. e. $\delta(n) = \mathcal{O}(n^{-\kappa})$ with $\kappa > 1$, so that $\operatorname{diam}\zeta \leq \delta(n)$ for all n-cylinder ζ and all n and ω.
Finally, we need some information on the measures:
(V) There exist $0 < d_0 < d_1$ and K such that $\rho^{d_0} \geq \mu(B_\rho) \geq \rho^{d_1}$ and

$$\frac{1}{K} \leq \frac{\mu(B_\rho)}{\mu^\omega(B_\rho)} \leq K$$

for all $\rho > 0$ small enough and for ν-almost every ω.
(VI) (Annulus condition) Assume that for some $\xi \geq \beta > 0$:

$$\sup_\omega \frac{\mu^\omega(B_{\rho+r} \setminus B_{\rho-r})}{\mu(B_\rho)} = \mathcal{O}(\frac{r^\xi}{\rho^\beta})$$

for every $r < \rho$.

Our main result is on the distribution of the first hitting and return times. For a set $B \subset M$ and $\omega \in \Omega$, one defines the function

$$\tau_B^\omega(x) = \inf\{j \geq 1 : \; T_\omega^j x \in B\}.$$

This is the *hitting time function* on M or the *return time function* when restricted to B itself. We can now state our main results, here μ_B^ω is the conditional measure of μ^ω restricted to the set $B \subset M$. First of all, under the previous assumption we obtain an exponential law for the distribution of hitting times.

Theorem 2.1. *Let a random dynamical system satisfy the above requirements (I)–(VI) where δ and λ both decay super-polynomially fast.*
Then

$$\mu^\omega\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

for all $t > 0$, for μ^ω-almost every $\mathsf{x} \in M$ and ν-almost every $\omega \in \Omega$.

Under the same assumptions, we can also prove an exponential law for the distribution of the return times. It is important to notice that this is the first paper where a quenched law is proved for the return times. Indeed, in [24, 25, 11], the law was only obtained for the hitting times. This is a significant difference with the deterministic setting where if a limiting distribution exists for the hitting times, then it also exists for the return times (and the other way around) [14].

Theorem 2.2. *Let a random dynamical system satisfy the above requirements (I)–(VI) where δ and λ both decay super-polynomially fast.*

Then

$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

for all $t > 0$*, for* μ^ω*-almost every* $\mathsf{x} \in M$ *and* ν*-almost every* $\omega \in \Omega$*.*

As it can be observed in the next theorem, even if δ and λ do not decay super-polynomially fast, one can still obtain an exponential distribution assuming some technical conditions on the constants present in the hypothesis (I)–(VI).

Let us denote u_0 the largest of the elements $\tilde{u}$ so that $\mu^\omega_{\gamma^u}(B_\rho(x)) \le C_1\rho^{\tilde{u}}$ for all $\rho > 0$ small enough and for almost all $x \in \gamma^u$, every unstable leaf γ^u and ν-almost all ω. By assumption (V), such element exists and is at least equal to d_0.

We have a version of Theorems 2.1 and 2.2 for polynomial decay:

Theorem 2.3. *Let a random dynamical system satisfy the above requirements (I)–(VI). Assume one of the following two conditions is satisfied.*
(A) $\delta(n) = \mathcal{O}(n^{-\kappa})$ *and* $\lambda(k) = \mathcal{O}(k^{-p})$ *decay polynomially with the respective rates* $\kappa > 1$ *and* $p > 1$ *satisfying* $\kappa\xi > 1$, $\max\{\frac{d_1\beta}{\kappa\xi - 1}, \left(\frac{\beta}{\xi} + d_1\right)\frac{1}{p}\} < \min\{1, u_0\}$ *and* $\gamma = \kappa u_0 - 2 - \kappa' > 1$*.*
(B) δ *decays super-polynomially,* $\lambda(k) = \mathcal{O}(k^{-p})$ *decays polynomially and* $\left(\frac{\beta}{\xi} + d_1\right)\frac{1}{p} < \min\{1, u_0\}$*.*

Then the conclusions of Theorems 2.1 and 2.2 hold.

Remark: If d is the dimension of the measure μ then $d_0 < d < d_1$ can be chosen arbitrarily close to d. The assumptions in case (A) then simplify to $\max(\frac{d\beta}{k\xi-1}, \frac{\beta/\xi+d}{p}) < 1 \wedge u_0$.

The proof of the theorems is done in the next three sections. In Sections 4 and 5 we prove that the limiting distribution is exponential (the convergence is realised for μ^ω-almost every point x) using a key proposition (Proposition 3.1). In Section 3, we prove the key proposition, i.e. we show that the measure of the set of points whose neighbourhoods return to themselves within a very small number of iterates is small. In Section 6 we then prove the limiting result for return times. Finally, in Section 7 we give examples to apply our main result.

Throughout the paper $C_0, C_1, \dots$ and $\alpha, \beta, \dots$ denote global constants while $c_0, c_1, \dots$ are locally defined constants.

3. Very Short Returns

For a ball $B_\rho(\mathsf{x}) \subset M$ we define the counting function

$$Z^\omega_{\mathsf{x},\rho,t}(y) = \sum_{n=0}^{\lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor - 1} \mathbb{1}_{B_\rho(\mathsf{x})} \circ T^n_\omega(y)$$

which tracks the number of visits a trajectory of the point $y \in M$ makes to the ball $B_\rho(\mathsf{x})$ on an orbit segment of length $N = \lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor$, where t is a positive parameter. Clearly $\tau^\omega_{B_\rho(\mathsf{x})}(y) > N$ exactly if $Z^\omega_{\mathsf{x},\rho,t}(y) = 0$.

Let us put $J = \mathfrak{a}|\log\rho|$ (with the number $\mathfrak{a}$ determined below) and define the following counting function for very short returns along the orbit segment:

$$Y^\omega_{\mathsf{x},\rho,t}(y) = \sum_{j=1}^{N} \mathbb{1}_{B_\rho(\mathsf{x})\cap\{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})}<J\}} \circ T^j_\omega(y).$$

In order to control the contribution made by the terms involving $Y^\omega_{\mathsf{x},\rho,t}$ we first have to consider the set of points that have very short returns. For that purpose, for a positive parameter $\mathfrak{a}$ let us define the set

$$\mathcal{V}^\omega_\rho(\mathfrak{a}) = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < \mathfrak{a}|\log\rho|\},$$

where $\rho > 0$. The set $\mathcal{V}^\omega_\rho$ represents the points within M with very short return times with respect to the realisation ω. The following of this section is dedicated to estimate the size of the set $\mathcal{V}^\omega_\rho$ and as a consequence to estimate the quantity $\mu^\omega(Y^\omega_{\mathsf{x},\rho,t})$.

Let us define $\mathfrak{a} = (4\log A)^{-1}$ with

$$A = \sup_\omega \left(\| DT_\omega \|_{\mathscr{L}^\infty} + \| DT^{-1}_\omega \|_{\mathscr{L}^\infty}\right) \tag{1}$$

$(A \geq 2)$. Then the set $\mathcal{V}_\rho \subset M$ consists of all $\mathsf{x} \in M$ for which $B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing$ for some $1 \leq n < J$.

3.1. **Estimate on the measure of $\mathcal{V}_\rho$.** Now we can show that the set of centres where small balls have very short returns is small. Even if we follow the proof of Proposition 5.1 of [15] which is modelled after Lemma 4.1 of [10], we emphasize that this is not a direct adaptation, in particular in view of Lemma 3.1 and its differences with the deterministic version.

Proposition 3.1. *There exist a constant $C_2 > 0$ such that for all ρ small enough and all ω and $\hat\omega$:*

$$\mu^{\hat\omega}(\mathcal{V}^\omega_\rho) \leq C_2\Big(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}\Big)$$

where $u_1 = u_0$ if δ decays superpolynomially and $u_1 = u_0 - \frac{1}{\kappa}$ if δ decays polynomially with power κ and $\mathfrak{b}, \mathfrak{c} > 0$ (recall that $\Theta(n) = \mathcal{O}(n^{\kappa'})$).

Proof. We partition $\mathcal{V}^\omega_\rho$ into level sets $\mathcal{N}^\omega_\rho(n)$ as follows

$$\mathcal{V}^\omega_\rho = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing \text{ for some } 1 \leq n < J\} = \bigcup_{n=1}^{J-1} \mathcal{N}^\omega_\rho(n)$$

where

$$\mathcal{N}^\omega_\rho(n) = \{\mathsf{x} \in M : B_\rho(\mathsf{x}) \cap T^{-n}_\omega B_\rho(\mathsf{x}) \neq \varnothing\}.$$

The above union is split into two collections $\mathcal{V}^{\omega,1}_\rho$ and $\mathcal{V}^{\omega,2}_\rho$, where

$$\mathcal{V}^{\omega,1}_\rho = \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}^\omega_\rho(n) \quad \text{and} \quad \mathcal{V}^{\omega,2}_\rho = \bigcup_{n=\lceil \mathfrak{b}J \rceil}^{J-1} \mathcal{N}^\omega_\rho(n)$$

and where the constant $\mathfrak{b} \in (0,1)$ will be chosen below. In order to find the measure of the total set we will estimate the measures of the two parts separately.

(I) Estimate of $\mathcal{V}^{\omega,2}_\rho$

We will derive a uniform estimate for the measure of the level sets $\mathcal{N}^\omega_\rho(n)$ when $n > \mathfrak{b}J$. For this purpose let $\tilde\omega$ be such that $\tilde\omega_i = \omega_i$ for all $0 \le i \le n-1$ and $\tilde\omega_i = \hat\omega_{i-n}$ for all $i \ge n$.

We have $T^n_{\tilde\omega} = T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0} = T^n_\omega$ and thus

$$\mu^{\hat\omega}(\mathcal{N}^\omega_\rho(n)) = \mu^{\tilde\omega}(T^{-n}_{\tilde\omega}\mathcal{N}^\omega_\rho(n)) \le \sum_\zeta \mu^{\tilde\omega}(T^{-n}_\omega \mathcal{N}^\omega_\rho(n) \cap \zeta)$$

where the sum is over all n-cylinders ζ. We will consider each of the measures $\mu^{\tilde\omega}(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap \zeta)$ separately by using the product form of the measures $\mu^{\hat\omega}$. By distortion of the Jacobian we obtain

$$\begin{aligned} \mu^{\tilde\omega}_{\gamma^u}(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta) &= \frac{\mu^{\tilde\omega}_{\gamma^u}(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta)}{\mu^{\tilde\omega}_{\gamma^u}(\zeta)}\,\mu^{\tilde\omega}_{\gamma^u}(\zeta) \\ &\le \Theta(n)\,\frac{\mu^{\hat\omega}_{\hat\gamma^u}(T^n_\omega(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta))}{\mu^{\hat\omega}_{\hat\gamma^u}(T^n_\omega\zeta)}\,\mu^{\tilde\omega}_{\gamma^u}(\zeta), \end{aligned} \tag{2}$$

where, as before, $\hat\gamma^u = \gamma^u(T^n_\omega x)$ for $x \in \zeta \cap \gamma^u$. We estimate the numerator by finding a bound for the diameter of the set. Let the points x and z in $T^{-n}_\omega\mathcal{N}^\omega_\rho(n)$ be such that $x, z \in T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta\cap\gamma^u$ for an unstable leaf γ^u.

Note that $T^n_\omega x, T^n_\omega z \in \mathcal{N}^\omega_\rho(n)$, there exists $y \in B_\rho(T^n_\omega x)$ such that $T^n_\omega y \in B_\rho(T^n_\omega x)$ (as $B_\rho(T^n_\omega x) \cap T^{-n}_\omega B_\rho(T^n_\omega x) \ne \emptyset$), thus

$$d(T^n_\omega x, x) \le d(T^n_\omega x, T^n_\omega y) + d(T^n_\omega y, y) + d(y,x) \le \rho + 2\rho + A^n d(T^n_\omega x, T^n_\omega y) \le (3+A^n)\rho$$

with A defined in (1). Hence as $y \in B_{A^n\rho}(x)$:

$$d(T^n_\omega x, T^n_\omega z) \le d(T^n_\omega x, x) + d(x,z) + d(z, T^n_\omega z) \le 6A^n\rho + d(x,z).$$

We have

$$d(x,z) \le \operatorname{diam}\zeta < \delta(n)$$

by assumption. Therefore

$$d(T^n_\omega x, T^n_\omega z) \le 6A^n\rho + d(x,z) \le 6\,A^n\rho + \delta(n).$$

If we choose $\mathfrak{a} > 0$ so that $\mathfrak{a} < \frac{1}{2\log A}$ then $A^n\rho < e^{-\frac12|\log\rho|^{1/2}}$. If $n \ge \mathfrak{b}|\log\rho|$ for some $\mathfrak{b} \in (0,\mathfrak{a})$ then

$$d(T^n_\omega x, T^n_\omega z) \le c_1(e^{-\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n))$$

for some constant c_1 where $\mathfrak{c}' = \min(\frac12, \sqrt{\mathfrak{b}})$. Taking the supremum over all points x and z yields

$$|\,T^n_\omega(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta\cap\gamma^u)\,| \le c_1(e^{-\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n)).$$

By assumption (V) on the relationship between the measure and the metric

$$\mu^{\hat\omega}_{\hat\gamma^u}(T^n_\omega(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta)) \le c_2(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n)^{u_0}).$$

Incorporating the estimate into (2) yields

$$\mu^{\tilde\omega}_{\gamma^u}(T^{-n}_\omega\mathcal{N}^\omega_\rho(n)\cap\zeta) \le c_4\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}} + \delta(n)^{u_0})\mu^{\tilde\omega}(\zeta),$$

for some c_4. Integrating over $dv(\gamma^u)$ and summing over ζ yields

$$\mu^{\hat\omega}(\mathcal{N}_\rho^\omega(n)) \le c_4\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\delta(n)^{u_0})\sum_\zeta \mu^{\tilde\omega}(\zeta) \le c_5\Theta(n)\,(e^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\delta(n)^{u_0})$$

as $\sum_\zeta \mu^{\tilde\omega}(\zeta)=\mathcal{O}(1)$. Consequently

$$\begin{aligned}
\mu^{\hat\omega}(\mathcal{V}_\rho^{\omega,2}) &\le \sum_{n=\lceil \mathfrak{b}J\rceil}^{J-1}\mu^{\hat\omega}(\mathcal{N}_\rho^\omega(n))\\
&\le c_5\Theta(J)\,Je^{-u_0\mathfrak{c}'|\log\rho|^{1/2}}+\sum_{n=\lceil \mathfrak{b}J\rceil}^{J-1}\Theta(n)\delta(n)^{u_0}\\
&\le c_6(e^{-\mathfrak{c}''|\log\rho|^{1/2}}+\delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}) \qquad (3)
\end{aligned}$$

for some constant $\mathfrak{c}''>0$ (and ρ small enough) as $J=\lfloor\mathfrak{a}|\log\rho|\rfloor$. Here $u_1\le u_0$ is so that $\sum_{n=n_0}^\infty\delta(n)^{u_0}\le c_7\delta(n_0)^{u_1}$ for some constant c_7.

(II) Estimate of $\mathcal{V}_\rho^{\omega,1}$

We will need the following randomised version of Lemma B.3 from [10].

Lemma 3.1. *Put $s_{p,k}=2^p\,\frac{A^{k\,2^p}-1}{A^k-1}$. Then for every p,k integers, $\rho>0$ and ω there exists an $\tilde\omega$ so that*

$$\left\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})\ne\varnothing\right\}\subset\left\{\mathsf{x}\in M: B_{s_{p,k}\rho}(\mathsf{x})\cap T_{\tilde\omega}^{-k2^p}B_{s_{p,k}\rho}(\mathsf{x})\ne\varnothing\right\}.$$

Proof. Consider the case $p=1$. Let x be such that $B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})\ne\varnothing$. This implies that there exists $z\in B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})$. Write $\varphi\in\mathscr{I}_k^\omega$ the inverse branch of T_ω^k such that $z=\varphi(T_\omega^k(z))$.

For any $u\in\varphi(B_\rho(\mathsf{x}))$, there exists $v\in B_\rho(\mathsf{x})$ such that $\varphi(v)=u$. Also note that $T_\omega^kz\in B_\rho(\mathsf{x})$, thus

$$d(u,\mathsf{x})\le d(u,z)+d(z,\mathsf{x})\le d(\varphi(v),\varphi(T_\omega^kz))+2\rho\le(2A^k+2)\rho.$$

Therefore, $\varphi(B_\rho(\mathsf{x}))\subset B_{(2A^k+2)\rho}(\mathsf{x})$.

One can observe that if $B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})\ne\varnothing$ then $\varphi\left(B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})\right)\ne\varnothing$ thus $\varphi(B_\rho(\mathsf{x}))\cap\varphi(T_\omega^{-k}B_\rho(\mathsf{x}))\ne\varnothing$ and therefore $B_{(2A^k+2)\rho}(\mathsf{x})\cap T_\omega^{-k}(T_\omega^{-k}B_{(2A^k+2)\rho}(\mathsf{x}))\ne\varnothing$. Finally, this gives us

$$\{\mathsf{x}\in M: B_\rho(\mathsf{x})\cap T_\omega^{-k}B_\rho(\mathsf{x})\ne\varnothing\}\subset\{\mathsf{x}\in M: B_{(2A^k+2)\rho}(\mathsf{x})\cap T_\omega^{-k}(T_\omega^{-k}B_{(2A^k+2)\rho}(\mathsf{x}))\ne\varnothing\}. \qquad (4)$$

If we choose $\tilde\omega$ such that $\tilde\omega_i=\omega_i$ for all $0\le i\le k-1$ and $\tilde\omega_i=\omega_{i-k}$ for all $i\ge k$ then $T_\omega^k\circ T_\omega^k=T_{\tilde\omega}^{2k}$. This proves the case $p=1$.

Since (4) is satisfied for every p,k integers, $\rho>0$ and ω we obtain by induction that

$$\begin{aligned}
&\{\mathsf{x}\in M: B_{(2A^k+2)\rho}(\mathsf{x})\cap T_{\tilde\omega}^{-2k}B_{(2A^k+2)\rho}(\mathsf{x})\ne\varnothing\}\\
\subset\;&\{\mathsf{x}\in M: B_{(2A^{2k}+2)(2A^k+2)\rho}(\mathsf{x})\cap T_{\tilde\omega}^{-2k}(T_{\tilde\omega}^{-2k}B_{(2A^{2k}+2)(2A^k+2)\rho}(\mathsf{x}))\ne\varnothing\}\\
=\;&\{\mathsf{x}\in M: B_{s_{2,k}\rho}(\mathsf{x})\cap T_{\bar\omega}^{-4k}B_{s_{2,k}\rho}(\mathsf{x})\ne\varnothing\}
\end{aligned}$$

with $\bar{\omega}$ such that $\bar{\omega}_i = \tilde{\omega}_i$ for all $0 \leq i \leq 2k-1$ and $\bar{\omega}_i = \tilde{\omega}_{i-2k}$ for all $i \geq 2k$ and this proves the case $p = 2$.

The general case is shown similarly by induction.

□

Let us now consider the case $1 \leq n \leq \lfloor \mathfrak{b}J \rfloor$ and let as in Lemma 3.1 $s_{p,n} = 2^p \frac{A^{n\,2^p}-1}{A^n-1}$. Hence by Lemma 3.1 one has $\mathcal{N}_\rho^\omega(n) \subset \mathcal{N}_{s_{p,n}\rho}^{\tilde{\omega}}(2^p n)$, where $\tilde{\omega} = \tilde{\omega}(n)$ depends on n, for any $p \geq 1$, and in particular for $p(n) = \lfloor \log \mathfrak{b}J - \lg n \rfloor + 1$. Therefore

$$\bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_\rho^\omega(n) \subset \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_{s_{p(n),n}\rho}^{\tilde{\omega}}(2^{p(n)}n).$$

Now define

$$n' = n2^{p(n)} \qquad \text{and} \qquad \rho' = s_{p(n),n}\rho.$$

A direct computation shows that $1 \leq n \leq \lfloor \mathfrak{b}J \rfloor$ implies $\lceil \mathfrak{b}J \rceil \leq n' \leq 2\mathfrak{b}J$ and so

$$\mathcal{V}_\rho^{\omega,1} = \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_\rho^\omega(n) \subset \bigcup_{n=1}^{\lfloor \mathfrak{b}J \rfloor} \mathcal{N}_{s_{p(n),n}\rho}^{\tilde{\omega}}(2^{p(n)}n) \subset \bigcup_{n'=\lceil \mathfrak{b}J \rceil}^{2\mathfrak{b}J} \mathcal{N}_{\rho'}^{\tilde{\omega}}(n').$$

Therefore, to estimate the measure of $\mathcal{V}_\rho^1$ it suffices to find a bound for $\mu^{\hat{\omega}}(\mathcal{N}_{\rho'}^{\tilde{\omega}}(n'))$ when $n' \geq \mathfrak{b}J$. This is accomplished by using an argument analogous to the first part of the proof. Notice that since $\tilde{\omega} = (\omega_0 \dots \omega_{k-1})^{2^p-1}\omega$ we get that

$$T_{\tilde{\omega}}^{n'} = (T_\omega^n)^{2^p}.$$

Define $\omega' = \omega_0 \dots \omega_{k-1}\tilde{\omega} = (\omega_0 \dots \omega_{k-1})^{2^p}\omega$ and $\hat{\omega}' = (\omega_0 \dots \omega_{k-1})^{2^p}\hat{\omega}$. Notice that $\theta^{2^p n}\omega' = \theta^{(2^p-1)n}\tilde{\omega} = \omega$, we get

$$T_{\omega'}^{n'} = (T_\omega^n)^{2^p} = T_{\tilde{\omega}}^{n'}.$$

As a result,

$$\mathcal{N}_{\rho'}^{\tilde{\omega}}(n') = \{x : B_{\rho'} \cap T_{\tilde{\omega}}^{-n'} B_\rho \neq \emptyset\} = \{x : B_{\rho'} \cap T_{\omega'}^{-n'} B_\rho \neq \emptyset\} = \mathcal{N}_{\rho'}^{\omega'}(n').$$

Similarly to the part (I),

$$\begin{aligned} \mu^{\hat{\omega}}(\mathcal{N}_{\rho'}^{\tilde{\omega}}(n')) &= \mu^{\hat{\omega}}(\mathcal{N}_{\rho'}^{\omega'}(n')) \\ &= \mu_{\gamma^u}^{\hat{\omega}'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta) \\ &= \frac{\mu_{\gamma^u}^{\hat{\omega}'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta)}{\mu_{\gamma^u}^{\hat{\omega}'}(\zeta)}\, \mu_{\gamma^u}^{\hat{\omega}'}(\zeta) \\ &\leq \Theta(n')\, \frac{\mu_{\hat{\gamma}^u}^{\hat{\omega}}(T_{\omega'}^{n'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta))}{\mu_{\hat{\gamma}^u}^{\hat{\omega}}(T_{\omega'}^{n'}\zeta)}\, \mu_{\gamma^u}^{\hat{\omega}'}(\zeta). \end{aligned}$$

To estimate the measure of the numerator we follow the proof of Proposition 3.1 and replace all the n with n' and ρ with ρ'. We get for $\mathfrak{b} < 1/3$

$$\operatorname{diam}(T_{\omega'}^{n'}(T_{\omega'}^{-n'}\mathcal{N}_{\rho'}^{\omega'}(n') \cap \zeta)) \leq c_1(e^{-\mathfrak{c}'|\log \rho'|^{1/2}} + \delta(n')).$$

Therefore

$$\mu^{\hat{\omega}}(\mathcal{N}_{\rho'}^{\tilde{\omega}}(n')) \leq c_5\Theta(n')(e^{-u_0|\log \rho'|^{1/2}} + \delta(n')^{u_0}).$$

Since $\rho' = s_{p,n}\rho$ and $\mathfrak{b} < \mathfrak{a} = \frac{1}{4\log A}$, we have

$$\begin{aligned}\rho' &\leq A^{2n2^p}\rho = A^{2n'}\rho \\ &\leq A^{4\mathfrak{b}J} = A^{4\mathfrak{a}\mathfrak{b}|\log\rho|}\rho \\ &\leq A^{\frac{4}{16\log A}|\log\rho|}\rho = \rho^{3/4}\end{aligned}$$

which gives us

$$\mu^{\hat{\omega}}(\mathcal{N}^{\tilde{\omega}}_{\rho'}(n')) \leq c_5\Theta(n')(e^{-u_0|\log\rho^{3/4}|^{1/2}} + \delta(n')^{u_0}).$$

Thus, we obtain an estimate similar to (3):

$$\mu^{\hat{\omega}}(\mathcal{V}^{\omega,1}_\rho) \leq \sum_{n'=\lceil \mathfrak{b}J\rceil}^{2\mathfrak{b}J} \mu^{\hat{\omega}}(\mathcal{N}^{\tilde{\omega}}_{\rho'}(n')) \leq c_8(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'})$$

for some $\mathfrak{c} \in (0, \frac{3}{4}u_0)$.

(III) Final estimate

Overall we obtain for all ρ sufficiently small

$$\mu^{\hat{\omega}}(\mathcal{V}^{\omega}_\rho) \leq \mu^{\hat{\omega}}(\mathcal{V}^{\omega,1}_\rho) + \mu^{\hat{\omega}}(\mathcal{V}^{\omega,2}_\rho) \leq C_2\,(e^{-\mathfrak{c}|\log\rho|^{1/2}} + \delta(\mathfrak{a}\mathfrak{b}|\log\rho|)^{u_1}|\log\rho|^{\kappa'}),$$

for some C_2. □

3.2. **Estimate of $\mu^\omega(Y^\omega)$.** Now we are in a position to estimate the μ^ω-measure of the function $Y^\omega_{\mathsf{x},\rho,t}$.

Lemma 3.2. *Put $\gamma = \kappa u_0 - 2 - \kappa'$. Then for any $\gamma' \in (0,\gamma)$ there exists a set $\mathcal{B}^\omega_\rho$ so that*

$$\mu^\omega(Y^\omega_{\mathsf{x},\rho,t}) \leq |\log\rho|^{-\gamma'}$$

for all $\mathsf{x} \notin \mathcal{B}^\omega_\rho$ and all $t > 0$ and

$$\mu^\omega(\mathcal{B}^\omega_\rho) \lesssim |\log\rho|^{-(\gamma-\gamma')}.$$

Proof. In order to estimate the term $\mu^\omega(Y^\omega_{\mathsf{x},\rho,t})$ let us put $N_\rho(\mathsf{x}) = \lfloor t/\mu(B_\rho(\mathsf{x}))\rfloor$. Observe that if $B_\rho(\mathsf{x})\cap\{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})} < J\} \neq \varnothing$ then there exist $y \in B_\rho(\mathsf{x})$ and $k < j$ so that $T^k_{\theta^j\omega}y \in B_\rho(\mathsf{x})$ which implies that $B_\rho(\mathsf{x})\cap T^{-k}_{\theta^j\omega}B_\rho(\mathsf{x}) \neq \varnothing$ and therefore $\mathsf{x} \in \mathcal{V}^{\theta^j\omega}_\rho$. Hence $\mathsf{x} \notin \mathcal{V}^{\theta^j\omega}_\rho$ implies $B_\rho(\mathsf{x})\cap\{\tau^{\theta^j\omega}_{B_\rho(\mathsf{x})} < J\} = \varnothing$.

Put $W^\omega_\rho(\mathsf{x}) = \sum_{j=1}^{N_\rho(\mathsf{x})} \mathbb{1}_{\mathcal{V}^{\theta^j\omega}_\rho}(\mathsf{x})$ and $q^\omega_\rho(\mathsf{x}) = \frac{W^\omega_\rho(\mathsf{x})}{N_\rho(\mathsf{x})}$. Let $M_k = \{\mathsf{x} \in M : N_\rho(\mathsf{x}) = k\}$ and put $a_{j,k} = \mu^\omega(\mathcal{V}^{\theta^j\omega}_\rho \cap M_k)$. Observe that, by Assumption (V) , $\sup_{\mathsf{x}} N^\omega_\rho(\mathsf{x})$ is bounded above by $\hat{N} = c_1 t\rho^{-d_1}$ for some constant c_1. Then

$$Q^\omega_\rho := \int_M q^\omega_\rho(\mathsf{x})\,d\mu^\omega(\mathsf{x}) = \sum_{k=1}^{\hat{N}}\frac{1}{k}\sum_{j=1}^{k} a_{j,k} \leq \sum_{j=1}^{\hat{N}}\frac{1}{j}\sum_{k=1}^{\hat{N}} a_{j,k}.$$

Since by Proposition 3.1 $\mu^\omega(\mathcal{V}^{\theta^j\omega}_\rho) = \sum_{k=1}^{\hat{N}} a_{j,k} \lesssim |\log\rho|^{-\gamma-1}$, where $\gamma = \kappa u_0 - 2 - \kappa'$, we thus obtain

$$Q^\omega_\rho \lesssim |\log\rho|^{-\gamma-1}\sum_{j=1}^{\hat{N}}\frac{1}{j} \lesssim |\log\rho|^{-\gamma}.$$

Now define

$$\mathcal{B}_\rho^\omega = \left\{ \mathsf{x} \in M : q_\rho^\omega(\mathsf{x}) > |\log \rho|^{-\gamma'} \right\}$$

for $\gamma' \in (0, \gamma)$. By Markov's inequality:

$$\mu^\omega(\mathcal{B}_\rho^\omega) \leq Q_\rho^\omega |\log \rho|^{\gamma'} \lesssim |\log \rho|^{-\gamma''},$$

where $\gamma'' = \gamma - \gamma'$.

If $\mathsf{x} \notin \mathcal{B}_\rho^\omega$ then $q_\rho^\omega(\mathsf{x}) \lesssim |\log \rho|^{-\gamma'}$ and $W_\rho^\omega(\mathsf{x}) \lesssim \frac{N_\rho(\mathsf{x})}{|\log \rho|^{\gamma'}}$. Consequently there exists an index set $\mathcal{I}_\mathsf{x}^\omega \subset \{1, 2, \ldots, N_\rho(\mathsf{x})\}$ so that $|\mathcal{I}_\mathsf{x}^\omega| \lesssim N_\rho(\mathsf{x}) |\log \rho|^{-\gamma'}$ and

$$\begin{cases} \mathsf{x} \in \mathcal{V}_\rho^{\theta^j \omega} & \forall j \in \mathcal{I}_\mathsf{x}^\omega \\ \mathsf{x} \notin \mathcal{V}_\rho^{\theta^j \omega} & \forall j \in \{1, \ldots, N_\rho(\mathsf{x})\} \setminus \mathcal{I}_\mathsf{x}^\omega. \end{cases}$$

Since $B_\rho(\mathsf{x}) \cap \{\tau_{B_\rho(\mathsf{x})}^{\theta^j \omega} < J\} = \varnothing$ for all $j \in \{1, \ldots, N_\rho(\mathsf{x})\} \setminus \mathcal{I}_\mathsf{x}^\omega$ we finally get by Assumption (V)

$$\begin{aligned} \mu^\omega(Y_{\mathsf{x},\rho,t}^\omega) &\leq \mu^\omega \left(\sum_{j \in \mathcal{I}_\mathsf{x}^\omega} \mathbb{1}_{B_\rho(\mathsf{x}) \cap \{\tau_{B_\rho(\mathsf{x})}^{\theta^j \omega} < J\}} \circ T_\omega^j \right) \\ &\leq |\mathcal{I}_\mathsf{x}^\omega| K \mu(B_\rho(\mathsf{x})) \\ &\lesssim q_{4\rho}^\omega(\mathsf{x}) N_\rho(\mathsf{x}) K \mu(B_\rho(\mathsf{x})) \\ &\lesssim \frac{t}{|\log \rho|^{\gamma'}} \end{aligned}$$

since $N_\rho(\mathsf{x}) = \lfloor t / \mu(B_\rho(\mathsf{x})) \rfloor$. □

4. Principal part

In this section we will look at the case when all returns within the observation time N are longer than J. In fact we want to show the following result which compares the hitting times distribution to that of independent random variables.

Proposition 4.1. *Under the assumptions of Theorem 2.1 put $u_1 = u_0$ if δ decays super-polynomially and $u_1 = u_0 - \frac{1}{\kappa}$ if $\delta(n) = \mathcal{O}(n^{-\kappa})$.*

Then there exist a positive ϵ and a positive constant C_3 so that

$$\left| \mu^\omega(\tau_{B_\rho}^\omega > N) - \prod_{j=1}^N (1 - \mu^{\theta^j \omega}(B_\rho)) \right| \leq C_3 \left(\rho^\epsilon + (\delta(J)^{u_1} + \rho^\epsilon) \sum_{j=1}^N \mu^{\theta^j \omega}(B_\rho) \right) + \mu^\omega(Y_{\mathsf{x},\rho,t}^\omega)$$

for all balls B_ρ and all $t > 0$.

Proof. We proceed as in [24, Lemma 6] and note that

$$\left| \mu^\omega(\tau_{B_\rho}^\omega > N) - \prod_{j=1}^N (1 - \mu^{\theta^j \omega}(B_\rho)) \right| \leq \sum_{j=1}^N \epsilon_{\theta^j \omega}(B_\rho) \prod_{k=1}^{j-1} (1 - \mu^{\theta^k \omega}(B_\rho))$$

where

$$\epsilon_\omega(B_\rho) = \sup_{k \geq 1} \left| \mu^\omega(\tau_{B_\rho}^\omega > k) \, \mu^\omega(B_\rho) - \mu^\omega(B_\rho \cap \{\tau_{B_\rho}^\omega > k\}) \right|. \tag{5}$$

Let $\Delta \leq N$ be an integer. As in [24, Lemma 8], we now split the error term on the RHS into three parts using the fact that

$$\epsilon_{\theta^j\omega} \leq \sup_{k\geq 1}\left|\mu^{\theta^j\omega}(B_\rho \cap T_{\theta^j\omega}^{-\Delta}\{\tau_{B_\rho}^{\theta^{j+\Delta}\omega} \geq k\}) - \mu^{\theta^j\omega}(B_\rho)\,\mu^{\theta^{j+\Delta}\omega}(\{\tau_{B_\rho}^{\theta^{j+\Delta}\omega} \geq k\})\right| + \mu^{\theta^j\omega}(B_\rho \cap \{\tau_{B_\rho}^{\theta^j\omega} \leq \Delta\}) + \mu^{\theta^j\omega}(B_\rho)\mu^{\theta^j\omega}(\tau_{B_\rho}^{\theta^j\omega} \leq \Delta).$$

Thus

$$\left|\mu^\omega(\tau_{B_\rho}^\omega > N) - \prod_{j=1}^{N}(1-\mu^{\theta^j\omega}(B_\rho))\right| \leq \sum_{j=1}^{N}\epsilon_{\theta^j\omega}(B_\rho) = \mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \tag{6}$$

where

$$\begin{aligned}
\mathcal{R}_1 &= \sum_{j=1}^{N}\sup_{k\geq 1}\left|\mu^{\theta^j\omega}(B_\rho \cap T_{\theta^j\omega}^{-\Delta}\{\tau_{B_\rho}^{\theta^{j+\Delta}\omega} \geq k\}) - \mu^{\theta^j\omega}(B_\rho)\,\mu^{\theta^{j+\Delta}\omega}(\{\tau_{B_\rho}^{\theta^{j+\Delta}\omega} \geq k\})\right| \\
\mathcal{R}_2 &= \sum_{j=1}^{N}\mu^{\theta^j\omega}(B_\rho \cap \{\tau_{B_\rho}^{\theta^j\omega} \leq \Delta\}) \\
\mathcal{R}_3 &= \sum_{j=1}^{N}\mu^{\theta^j\omega}(B_\rho)\mu^{\theta^j\omega}(\tau_{B_\rho}^{\theta^j\omega} \leq \Delta).
\end{aligned}$$

We now estimate the three terms individually and will choose the gap Δ carefully in Section 4.4 in order to use our mixing assumptions.

4.1. **Estimating $\mathcal{R}_1$.** We estimate the principal term by

$$\mathcal{R}_1 = N\sup_{\omega}\sup_{k\geq 1}\left|\mu^\omega(B_\rho \cap T_\omega^{-\Delta}S_k) - \mu^\omega(B_\rho)\,\mu^{\theta^\Delta\omega}(S_k)\right|$$

where $S_k = S_k(\Delta) = \{y : \tau_{B_\rho}^{\theta^\Delta\omega}(y) \geq k\}$. We now use the decay of correlations from Assumption (II) to obtain an estimate. Approximate $\mathbb{1}_{B_\rho}$ by Lipschitz functions from above and below as follows:

$$\phi(x) = \begin{cases} 1 & \text{on } B_\rho \\ 0 & \text{outside } B_{\rho+\delta\rho}\end{cases} \qquad \text{and} \qquad \tilde{\phi}(x) = \begin{cases} 1 & \text{on } B_{\rho-\delta\rho} \\ 0 & \text{outside } B_\rho\end{cases}$$

with both functions linear within the annuli. The Lipschitz norms of both ϕ and $\tilde{\phi}$ are equal to $1/\delta\rho$ and $\tilde{\phi} \leq \mathbb{1}_{B_\rho} \leq \phi$. We obtain

$$\begin{aligned}
&\mu^\omega(B_\rho \cap T_\omega^{-\Delta}S_k) - \mu^\omega(B_\rho)\,\mu^{\theta^\Delta\omega}(S_k) \\
&\leq \int_M \phi(\mathbb{1}_{S_k}\circ T_\omega^\Delta)\,d\mu^\omega - \int_M \mathbb{1}_{B_\rho}\,d\mu^\omega \int_M \mathbb{1}_{S_k}\,d\mu^{\theta^\Delta\omega} \\
&= X + Y
\end{aligned}$$

where

$$X = \left(\int_M \phi\, d\mu^\omega - \int_M \mathbb{1}_{B_\rho}\, d\mu^\omega\right) \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta\omega}$$

$$Y = \int_M \phi\,(\mathbb{1}_{S_k} \circ T_\omega^\Delta)\, d\mu^\omega - \int_M \phi\, d\mu^\omega \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta\omega}.$$

The two terms X and Y are estimated separately. The first term is estimated as follows:

$$X \le \int_M \mathbb{1}_{S_k}\, d\mu^{\theta^\Delta\omega} \int_M (\phi - \mathbb{1}_{B_\rho})\, d\mu^\omega \le \mu^\omega(B_{\rho+\delta\rho} \setminus B_\rho).$$

In order to estimate the second term Y we use the decay of correlations and have to approximate $\mathbb{1}_{S_k}$ by a function which is constant on local stable leaves. For that purpose put

$$\mathcal{S}_n = \bigcup_{\substack{\gamma^s \\ T_\omega^n\gamma^s \subset B_\rho}} T_\omega^n \gamma^s, \qquad \partial\mathcal{S}_n = \bigcup_{\substack{\gamma^s \\ T_\omega^n\gamma^s \cap B_\rho \neq \varnothing}} T_\omega^n \gamma^s$$

and

$$\mathscr{S}_\Delta^{N-j} = \bigcup_{n=\Delta}^{N-j} \mathcal{S}_n, \qquad \partial\mathscr{S}_\Delta^{n-j} = \bigcup_{n=\Delta}^{N-j} \partial\mathcal{S}_n.$$

The set

$$\mathscr{S}_\Delta^{N-j}(k) = S_k \cap \mathscr{S}_\Delta^{N-j}$$

is then a union of local stable leaves. This follows from the fact that by construction $T^n y \in B_\rho$ if and only if $T^n\gamma^s(y) \subset B_\rho$. We also have $S_k \subset \tilde{\mathscr{S}}_\Delta^{N-j}(k)$ where the set $\tilde{\mathscr{S}}_\Delta^{N-j}(k) = \mathscr{S}_\Delta^{N-j}(k) \cup \partial\mathscr{S}_\Delta^{N-j}$ is a union of local stable leaves.

Denote by ψ_Δ^{N-j} the indicator function of $\mathscr{S}_\Delta^{N-j}(k)$ and by $\tilde\psi_\Delta^{N-j}$ the indicator function of $\tilde{\mathscr{S}}_\Delta^{N-j}(k)$. Then ψ_Δ^{N-j} and $\tilde\psi_\Delta^{N-j}$ are constant on local stable leaves and satisfy

$$\psi_\Delta^{N-j} \le \mathbb{1}_{S_k} \le \tilde\psi_\Delta^{N-j}.$$

Since $\{y : \psi_\Delta^{N-j}(y) \neq \tilde\psi_\Delta^{N-j}(y)\} \subset \partial\mathscr{S}_\Delta^{N-j}$ we need to estimate the measure of $\partial\mathscr{S}_\Delta^{N-j}$.

By the contraction property $\operatorname{diam}(T_\omega^n\gamma^s(y)) \le \delta(n)$ and consequently

$$\bigcup_{\substack{\gamma^s \\ T_\omega^n\gamma^s \subset B_\rho}} T_\omega^n \gamma^s \subset B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}$$

and therefore

$$\mu^\omega(\partial\mathscr{S}_\Delta^{N-j}) \le \mu^\omega\left(\bigcup_{n=\Delta}^{N-j} T_\omega^{-n}\left(B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}\right)\right) \le \sum_{n=\Delta}^{N-j} \mu^{\theta^n\omega}(B_{\rho+\delta(n)} \setminus B_{\rho-\delta(n)}).$$

Hence, by assumption (VI), using $r = 2\delta(n) = \mathcal{O}(n^{-\kappa})$ if δ decays polynomially with power κ:

$$\begin{aligned}\sum_{n=\Delta}^{N-j} \mu^\omega(\partial\mathscr{S}_\Delta^{N-j}) &= \mathcal{O}(1) \sum_{n=\Delta}^{\infty} \frac{n^{-\kappa\xi}}{\rho^{d_1\beta}} \mu(B_\rho) \\ &= \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta}\mu(B_\rho))\end{aligned}$$

provided $\Delta \sim \rho^{-v}$ for some positive $v > \frac{\beta-d_0}{\kappa\xi-1}$ which is determined in Section 4.4 below. If we split $\Delta = \Delta' + \Delta''$ then we can estimate Y as follows:

$$\begin{aligned} Y &= \left| \int_M \phi \, T_\omega^{-\Delta'}(\mathbb{1}_{S_k(\Delta')}) \, d\mu^\omega - \int_M \phi \, d\mu^\omega \int_M \mathbb{1}_{S_k(\Delta)} \, d\mu^{\theta^\Delta \omega} \right| \\ &\leq \lambda(\Delta') \|\phi\|_{Lip} \|\mathbb{1}_{\tilde{\mathscr{S}}_{\Delta''}^{N-j-p'}}\|_{\mathscr{L}^\infty} + 2\mu^\omega(\partial \mathscr{S}_{\Delta''}^{N-j}). \end{aligned}$$

Hence

$$\mu^\omega(B_\rho \cap T^{-\Delta} S_k) - \mu^\omega(B_\rho) \, \mu^{\theta^\Delta \omega}(S_k) \leq \frac{\lambda(\Delta/2)}{\delta\rho} + \mu^\omega(B_\rho \setminus B_{\rho-\delta\rho}) + \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta} \mu(B_\rho)).$$

A similar estimate from below can be done using $\tilde{\phi}$. Hence

$$\mathcal{R}_1 \leq N c_1 \left(\frac{\lambda(\Delta/2)}{\delta\rho} + \sup_\omega \mu^\omega(B_{\rho+\delta\rho} \setminus B_{\rho-\delta\rho}) \right) + \mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta}).$$

4.2. **Estimating the terms $\mathcal{R}_2$.** We will estimate the measure of each of the summands comprising $\mathcal{R}_2$ individually. We use the product form of the measures μ^ω. For that purpose fix j and and let γ^u be an unstable local leaf through B. Then we put

$$\mathscr{C}_j^\omega(B, \gamma^u) = \{\zeta_{\varphi,j} : \zeta_{\varphi,j} \cap B \neq \varnothing, \varphi \in \mathscr{I}_j^\omega\}$$

for the cluster of j-cylinders that covers the set B, where the sets $\zeta_{\varphi,k}$ are the images of imbedded R-balls in $T_\omega^j \gamma^u$. Then, using the distortion property (III),

$$\begin{aligned} \mu_{\gamma^u}^\omega(T_\omega^{-j} B_\rho \cap B_\rho) &\leq \sum_{\zeta \in \mathscr{C}_j^\omega(B_\rho, \gamma^u)} \frac{\mu_{\gamma^u}^\omega(T_\omega^{-j} B_\rho \cap \zeta)}{\mu_{\gamma^u}^\omega(\zeta)} \mu_{\gamma^u}^\omega(\zeta) \\ &\leq \sum_{\zeta \in \mathscr{C}_j^\omega(B_\rho, \gamma^u)} \Theta(j) \frac{\mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(B_\rho \cap T_\omega^j \zeta)}{\mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(T_\omega^j \zeta)} \mu_{\gamma^u}^\omega(\zeta). \end{aligned}$$

Since $\mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(T_\omega^j \zeta) = \mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(B_{R,\gamma^u}(y_k))$ (for some y_k) is uniformly bounded from below, we obtain

$$\begin{aligned} \mu_{\gamma^u}^\omega(T_\omega^{-j} B_\rho \cap B_\rho) &\leq \Theta(j) \mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(B_\rho) \sum_{\zeta \in \mathscr{C}_j^\omega(B_\rho, \gamma^u)} \mu_{\gamma^u}^\omega(\zeta) \\ &\leq \Theta(j) \mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(B_\rho) \, L \, \mu_{\gamma^u}^\omega \left(\bigcup_{\zeta \in \mathscr{C}_j^\omega(B_\rho, \gamma^u)} \zeta \right). \end{aligned}$$

Now, since $\operatorname{diam} \bigcup_{\zeta \in \mathscr{C}_j^\omega(B_\rho, \gamma^u)} \zeta \leq \delta(j) + \operatorname{diam} B_\rho \leq c_1 \delta(j)$ (as we can assume that $\rho < \delta(j)$) we obtain

$$\mu_{\gamma^u}^\omega(T_\omega^{-j} B_\rho \cap B_\rho) \leq c_3 \Theta(j) \mu_{T_\omega^j \gamma^u}^{\theta^j \omega}(B_\rho) \delta(j)^{u_0}.$$

Since $d\mu^\omega = d\mu_{\gamma^u}^\omega d\upsilon^\omega(\gamma^u)$ we obtain

$$\mu^\omega(T_\omega^{-j} B_\rho \cap B_\rho) \leq c_4 \Theta(j) \mu^{\theta^j \omega}(B_\rho) \delta(j)^{u_0}.$$

Summing up the $\mu^\omega(T_\omega^{-j}B_\rho\cap B_\rho)$ over $j=J,\dots,\Delta-1$, we get

$$\begin{aligned}\mathcal{R}_2'(\omega) &= \mu^\omega(B_\rho\cap T_\omega^{-J}\{\tau_{B_\rho}^{\theta^J\omega}<\Delta-J\})+\mu^\omega(B_\rho\cap\{\tau_{B_\rho}^\omega<J\})\\ &\le \sum_{j=J}^{\Delta-1}\mu^\omega(T_\omega^{-j}B_\rho\cap B_\rho)+\mu^\omega(B_\rho\cap\{\tau_{B_\rho}^\omega<J\})\\ &\le c_4\sum_{j=J}^{\Delta-1}\Theta(j)\delta(j)^{u_0}\mu^{\theta^j\omega}(B_\rho)+\mu^\omega(B_\rho\cap\{\tau_{B_\rho}^\omega<J\}). \qquad (7)\end{aligned}$$

For the entire error term we thus obtain

$$\begin{aligned}\mathcal{R}_2 &= \sum_{k=1}^N\mathcal{R}_2'(\theta^k\omega)\\ &\le c_4\sum_{j=J}^{\Delta}\Theta(j)\delta(j)^{u_0}\sum_{k=1}^N\mu^{\theta^{k+j}\omega}(B_\rho)+\sum_{k=1}^N\mu^{\theta^k\omega}(B_\rho\cap\{\tau_{B_\rho}^{\theta^k\omega}<J\})\\ &\le c_6t\delta(J)^{u_1}J^{\kappa'}\sum_{k=1}^N\mu^{\theta^k\omega}(B_\rho)+\mu^\omega(Y_{\mathsf{x},\rho,t}^\omega)\end{aligned}$$

for some c_5,c_6 and almost every ω and ρ small enough (depending on ω). The exponent u_1 equals u_0 if $\delta(j)$ decays super-polynomially and equals $u_0-\frac{1}{\kappa}$ if $\delta(j)$ decays polynomially with power κ.

4.3. **Estimating the terms $\mathcal{R}_3$.**

Assumption (V) yields

$$\mu^\omega(B_\rho)=\int\mu_{\gamma^u}^\omega(B_\rho)\,dv^\omega(\gamma^u)\le\int C_1\rho^{u_0}\,dv^\omega(\gamma^u)=C_1\rho^{u_0}$$

for every ω. Since $\mu^{\theta^j\omega}(\tau_{B_\rho}^{\theta^j\omega}\le\Delta)\le\sum_{k=1}^{\Delta}\mu^{\theta^{j+k}\omega}(B_\rho)$ we obtain by Assumption (V)

$$\begin{aligned}\mathcal{R}_3 &= \sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)\mu^{\theta^j\omega}(\tau_{B_\rho}^{\theta^j\omega}\le\Delta)\\ &\le C_1\rho^{u_0}\sum_{j=1}^N\sum_{k=1}^{\Delta}\mu^{\theta^{j+k}\omega}(B_\rho)\\ &\le C_1\rho^{u_0}\Delta\sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)+C_1\rho^{u_0}\sum_{j=1}^{\Delta}(\Delta-j)\mu^{\theta^{N+j}\omega}(B_\rho)\\ &\le c_7\rho^{u_0}\Delta\sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)+c_7(\Delta\rho^{u_0})^2\end{aligned}$$

for some c_7 for almost all ω and ρ small enough since the first sum converges ν-almost everywhere to t.

4.4. **The total error.** The total error is

$$\begin{aligned}\mathcal{R} &= \mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3\\ &\le Nc_1\left(\frac{\lambda(\Delta)}{\delta\rho}+\sup_\omega \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right)+\mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta})\\ &\qquad +\left(c_6\delta(J)^{u_1}J^{\kappa'}+C_1\rho^{u_0}\Delta\right)\sum_{j=1}^N \mu^{\theta^j\omega}(B_\rho)+c_7(\Delta\rho^{u_0})^2+\mu^\omega(Y^\omega_{\mathsf{x},\rho,t}).\end{aligned}$$

Let us consider the case when λ decays polynomially with power p, i.e. $\lambda(k)\sim k^{-p}$. We can choose $\Delta=\rho^{-v}$ so that $\lambda(\Delta)=\mathcal{O}(\rho^{-vp})=\mathcal{O}(\rho^w\mu(B_\rho))\rho^{-(vp-w-d-1)}$ and so that for some $\epsilon>0$:

$$\Delta\rho^{u_0}\le c_8\rho^{u_0-\frac{w}{p}}\mu(B_\rho)^{-\frac1p}<c_9\rho^{u_0-\frac{w+d_1}{p}}=\mathcal{O}(\rho^\epsilon),$$

that is $\frac{d_1+w}{p}<v<\min\{1,u_0\}$. We then obtain with $\delta\rho=\rho^w$ that $N\frac{\lambda(\Delta)}{\delta\rho}\le c_{10}\frac{1}{\mu(B_\rho)}\frac{\Delta^{-p}}{\rho^w}=\mathcal{O}(\rho^{-(vp-w-d-1)})$. The second term is estimated by (maybe for some smaller $\epsilon>0$)

$$\sup_\omega \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})=\mathcal{O}(\frac{\rho^{w\xi}}{\rho^\beta})=\mathcal{O}(\rho^{w\xi-\beta})=\mathcal{O}(\rho^\epsilon)$$

since $w\xi>\beta$. Hence we need constants $w,v>0$ such that the following inequalities hold:

(1) $\frac{d_1+w}{p}<v<\min\{1,u_0\}$;
(2) $v(\kappa\xi-1)-d_1\beta>0$ from Section 4.1; and
(3) $w<\frac{\xi}{\beta}$ (w can be arbitrarily close to $\frac{\xi}{\beta}$).

These conditions hold if we require that $\max\{\frac{d_1\beta}{\kappa\xi-1},\left(\frac{\beta}{\xi}+d_1\right)\frac1p\}<\min\{1,u_0\}$ which in particular implies $\rho^{v(\kappa\xi-1)-d_1\beta}=\mathcal{O}(\rho^\epsilon)$. We therefore obtain

$$\mathcal{R}\le\mathcal{O}(\rho^\epsilon)+\mathcal{O}(J^{\kappa'}\delta(J)^{u_1}+\rho^\epsilon)\sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)+\mu^\omega(Y^\omega_{\mathsf{x},\rho,t})$$

for all ρ small enough and every x. □

5. Proof of the main theorems for hitting times

Proof of Theorem 2.1 and Theorem 2.3 for hitting times. According to [24] Lemma 14 the variance (as a function of ω) of

$$\mu^\omega(Z^\omega_{\mathsf{x},\rho,t})=\sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)$$

is bounded by ρ^q for some $0<q<\frac{(p-1)d_0\xi-d_1\xi-\beta}{p\xi+1}$ and for all x. Hence we obtain along any sequence ρ_i for which $\sum_{i=1}^\infty\rho_i^q<\infty$ by an application of Chebycheff's inequality and the Borel-Cantelli lemma that

$$\mu^\omega(Z^\omega_{\mathsf{x},\rho_i,t})\to t\qquad\text{as } i\to\infty$$

for ν-almost every ω since $\mu(Z^\omega_{\mathsf{x},\rho,t}) = t$ for all $\rho > 0$. Then,

$$\begin{aligned}\prod_{j=1}^N(1-\mu^{\theta^j\omega}(B_\rho)) &= \exp\sum_{j=1}^N\left(\mu^{\theta^j\omega}(B_\rho)+\mathcal{O}(\mu^{\theta^j\omega}(B_\rho))^2\right)\\ &= \exp\sum_{j=1}^N\mu^{\theta^j\omega}(B_\rho)\left(1+\mathcal{O}(\max_j\mu^{\theta^j\omega}(B_\rho))\right)\\ &\longrightarrow e^{-t}\end{aligned}$$

as $\rho\to 0$ along a sequence ρ_i, since $\max_j\mu^{\theta^j\omega}(B_\rho)\le C_1\rho^{u_0}\to 0$.

Thus, our theorems are proven along a sequence ρ_i provided we prove that the last term on the right hand side of the inequality in Proposition 4.1 goes to zero as $\rho_i\to 0$. That is we must show that for almost every x and for almost every ω the quantity $\mu^\omega(Y^\omega_{\mathsf{x},\rho_i})$ goes to zero. For this purpose, we use Lemma 3.2 and let $\gamma''\in(0,\gamma)$ (which can be chosen arbitrarily close to γ). Let $\mathcal{B}^\omega_{\rho_i}$ be the set given by Lemma 3.2, we have $\mu^\omega(\mathcal{B}^\omega_{\rho_i})\lesssim|\log\rho|^{-\gamma''}$. Let $\alpha\in(\frac{1}{\gamma''},1)$ then put $\rho_i=e^{-i^\alpha}$ for all $i\in\mathbb{N}$ large enough. These choices imply that $\sum_{i=1}^\infty\rho_i^q<\infty$. Since

$$\mu^\omega(\mathcal{B}^\omega_{\rho_i})\le c_1 i^{-\alpha\gamma''}$$

we thus obtain $\sum_i\mu^\omega(\mathcal{B}^\omega_{\rho_i})<\infty$ (as $\alpha\gamma''>1$). By the Borel-Cantelli lemma we now conclude that $\mu^\omega(x\in\mathcal{B}^\omega_{\rho_i}\text{ i.o.})=0$ which implies that $\mu^\omega(Y^\omega_{\mathsf{x},\rho_i,t})\lesssim|\log\rho_i|^{-(\gamma-\gamma'')}\longrightarrow 0$ as $i\to\infty$ for almost every x and ω.

This concludes the proof of the two main theorems for hitting times along the sequence ρ_i.

In order to get the convergence for arbitrary $\rho \to 0$ let $\rho > 0$ be sufficiently small and i so that $\rho_i \leq \rho \leq \rho_{i-1}$. We have $r_i = \rho_{i-1} - \rho_i = \rho_i.\mathcal{O}(i^{-(1-\alpha)})$, then

$$\begin{aligned}
&\left|\mu^\omega\left(\tau^\omega_{B_\rho} > \frac{t}{\mu(B_\rho)}\right) - \mu^\omega\left(\tau^\omega_{B_{\rho_i}} > \frac{t}{\mu(B_{\rho_i})}\right)\right| \\
\leq\ & \mu^\omega\left(\tau^\omega_{B_\rho \setminus B_{\rho_i}} < \frac{t}{\mu(B_\rho)}\right) + \mu^\omega\left(\frac{t}{\mu(B_\rho)} < \tau^\omega_{B_{\rho_i}} < \frac{t}{\mu(B_{\rho_i})}\right) \\
\leq\ & \sum_{j=1}^{\frac{t}{\mu(B_\rho)}} \mu^{\theta^j\omega}(B_\rho \setminus B_{\rho_i}) + \sum_{j=\frac{t}{\mu(B_\rho)}}^{\frac{t}{\mu(B_{\rho_i})}} \mu^{\theta^j\omega}(B_{\rho_i}) \\
\leq\ & \mu(B_\rho) \sum_{j=1}^{\frac{t}{\mu(B_\rho)}} \frac{\mu^{\theta^j\omega}(B_{\rho+r_i} \setminus B_{\rho-r_i})}{\mu(B_\rho)} + K\mu(B_{\rho_i}) \left|\frac{t}{\mu(B_\rho)} - \frac{t}{\mu(B_{\rho_i})}\right| \\
\lesssim\ & t\frac{r_i^\xi}{\rho_i^\beta} + Kt\frac{\mu(B_\rho \setminus B_{\rho_i})}{\mu(B_\rho)} \\
\lesssim\ & t(1+K)\frac{r_i^\xi}{\rho_i^\beta} \\
\lesssim\ & \frac{\rho_i^{\xi-\beta}}{i^{\xi(1-\alpha)}}
\end{aligned}$$

using Assumption (VI) and Assumption (V). This difference goes to zero as $i \to \infty$ since $\xi \geq \beta$ and $1 - \alpha > 0$ which concludes the proof of the theorem. $\square$

6. Return times distribution

Proof of Theorem 2.2 and Theorem 2.3 for return times. Since we proved an exponential distribution for the hitting times, to get an exponential distribution for the return times we will estimate the difference between the hitting time statistics and the return time statistics. To do so, we use the definition of ϵ_ω in (5) to notice that

$$\left|\mu^\omega_{B_\rho}(\tau^\omega_{B_\rho} > N) - \mu^\omega(\tau^\omega_{B_\rho} > N)\right| \leq \frac{\epsilon_\omega(B_\rho)}{\mu^\omega(B_\rho)}. \tag{8}$$

Observe that the error term in the RHS is different from the error term in (6), thus we cannot control this error term immediately using Section 4, as one could have hoped for from the deterministic case (e.g. [26]) or the annealed case [23].

In order to estimate the error term we split the RHS in three terms as in (6):

$$\left|\frac{\epsilon_\omega(B_\rho)}{\mu^\omega(B_\rho)}\right| = \tilde{\mathcal{R}} \leq \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 + \tilde{\mathcal{R}}_3.$$

We estimate the first term $\tilde{\mathcal{R}}_1$, the decay of correlations term (unlike the term $\mathcal{R}_1$ in Theorem 2.1 there is no factor N):

$$\tilde{\mathcal{R}}_1 = \frac{1}{\mu^\omega(B_\rho)} \sup_{k\geq 1} \left|\mu^\omega(B_\rho \cap T_\omega^{-\Delta} S_k) - \mu^\omega(B_\rho)\, \mu^{\theta^\Delta\omega}(S_k)\right|.$$

As in Section 4.1, we get:

$$\tilde{\mathcal{R}}_1 \le \frac{c_1}{\mu^\omega(B_\rho)}\left(\frac{\lambda(\Delta/2)}{\delta\rho} + \mu^\omega(B_{\rho+\delta\rho}\setminus B_{\rho-\delta\rho})\right) + \frac{\mu(B_\rho)}{\mu^\omega(B_\rho)}\mathcal{O}(\rho^{v(\kappa\xi-1)-d_1\beta})$$

where we use Assumption (V) to estimate the term $\frac{\mu(B_\rho)}{\mu^\omega(B_\rho)}$ by K. The short hitting times term, $\tilde{\mathcal{R}}_3$, can be dealt with easily as in Section 4.3:

$$\tilde{\mathcal{R}}_3 = \mu^\omega(\{y : \tau^\omega_{B_\rho}(y) < \Delta\}) \le \sum_{k=1}^{\Delta}\mu^{\theta^k\omega}(B_\rho) \le c_2\Delta\rho^{u_0}.$$

We are left with the short return times term, $\tilde{\mathcal{R}}_2$ which we estimate using (7):

$$\tilde{\mathcal{R}}_2 = \frac{\mathcal{R}_2'(\omega)}{\mu^\omega(B_\rho)} = \frac{1}{\mu^\omega(B_\rho)}\mu^\omega(B_\rho\cap\{y : \tau^\omega_{B_\rho}(y) < \Delta\}) \le c_3\sum_{j=J}^{\Delta-1}\Theta(j)\delta(j)^{u_0}\frac{\mu^{\theta^j\omega}(B_\rho)}{\mu^\omega(B_\rho)}$$

for all $\mathsf{x} \notin \mathcal{V}^\omega_{4\rho}$ (since $B_\rho(\mathsf{x})\cap\{\tau^\omega_{B_\rho(\mathsf{x})} < J\} = \varnothing$ for such x). This implies by Assumption (V) $\tilde{\mathcal{R}}_2 \le c_4\delta(J)^{u_1}J^{\kappa'}$.

Consequently, proceeding as in Section 4.4 we finally obtain

$$\tilde{\mathcal{R}} \le c_5\Big(\delta(J)^{u_0}J^{\kappa'} + \rho^\epsilon\Big)$$

for some $\epsilon >$ and for all $\mathsf{x} \notin \mathcal{V}^\omega_{4\rho}$. □

7. Examples

7.1. **Random C^2 interval maps.** As an example we consider random maps on the unit interval I, following the work of Buzzi [9]. Unlike in his paper where continuous piecewise monotonic and non-singular (p.m.n.s.) maps were studied, here we will consider piecewise monotonic C^2 maps where the diameter of n-cylinders (as defined in Assumption IV) are contracted. This will allow us to obtain uniform constants in the Doeblin-Fortet inequality and hence in the quenched decay of correlations.

Let $S : \Omega \times I \circlearrowleft$ be a skew action where the map θ is acting invertibly on Ω. For each ω the map $T_\omega : I \to I$ is assumed to be a piecewise monotonic C^2 map on the interval I with finitely many monotonic intervals and uniformly bounded C^2 norms, though we do not assume that $\{T_\omega\}_\omega$ share the same monotonic intervals. Recall that $\mathscr{I}^\omega_n$ are the inverse branches of T^n_ω. For $\varphi \in \mathscr{I}^\omega_n$ we denote by $\zeta_\varphi = \varphi(I)$ the n-cylinder associated with φ as before, and put

$$\delta(n) = \sup_\omega \max_{\varphi\in\mathscr{I}^\omega_n} |\zeta_\varphi|.$$

Like in the main theorem, we will assume that $\delta(n)$ decreases to zero as $n \to \infty$.

For a function $\psi : I \to \mathbb{R}$ we denote by var ψ its variation on the unit interval and let

$$\|f\| = \text{var } \psi + \|f\|_{\mathscr{L}^1}$$

be its norm. This makes $X = \{f \in C(I,\mathbb{R}), \|f\| < \infty\}$ a Banach space which is equipped with the strong norm $\|\cdot\|$ and the weak norm $\|\cdot\|_{\mathscr{L}^1}$. Consider the transfer operator $\mathcal{L}$

on X which for each ω maps a function $\psi \in X$ on the interval to a function $\mathcal{L}_\omega\psi$ on the interval. It is given by

$$\mathcal{L}_\omega\psi(x) = \sum_{\varphi\in\mathscr{I}_1^\omega} \frac{\psi(\varphi x)}{|DT_\omega(\varphi x)|}.$$

The iterates of the transfer operator are $\mathcal{L}_\omega^n = \mathcal{L}_{\theta^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\theta\omega} \circ \mathcal{L}_\omega$. We shall next prove the Doeblin-Fortet inequality:

Lemma 7.1. *Assume that $\delta(k)$ decreases to zero as $k \to \infty$. Then there exist $\eta < 1$, $n \in \mathbb{N}$ and a constant C_4 so that for every $\omega \in \Omega$ and $\psi : I \to \mathbb{R}$ with $\|\psi\| < \infty$ one has*

$$\text{var } \mathcal{L}_\omega^n\psi \le \eta \text{ var } \psi + C_4\|\psi\|_{\mathscr{L}^1}.$$

Proof. Let us fix ω. In order to estimate var $|DT_\omega^n|^{-1}$ let $\ell \in \mathbb{N}$ and $n = p\ell$. Then, for $\varphi \in \mathscr{I}_n^\omega$, $|DT_\omega^{p\ell}|^{-1}\varphi = (|DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi)|DT_\omega^\ell|^{-1}\varphi$ and

$$\text{var } |DT_\omega^{p\ell}|^{-1}\varphi \le ||DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi|_\infty \text{ var } |DT_\omega^\ell|^{-1}\varphi + ||DT_\omega^\ell|^{-1}\varphi|_\infty \text{ var } |DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi.$$

There exists a constant c_1 so that

$$||DT_\omega^\ell|^{-1}\varphi|_\infty \le c_1|\zeta_\varphi| \le c_1\delta(\ell)$$

and similarly $|DT_\omega^{(p-1)\ell}|^{-1}\varphi|_\infty \le c_1\delta((p-1)\ell)$. Hence since $T_\omega^\ell\varphi \in \mathscr{I}_{(p-1)\ell}^{\theta^\ell\omega}$ one has

$$\text{var } |DT_\omega^{p\ell}|^{-1}\varphi \le c_1\delta(\ell) \text{ var } |DT_\omega^{(p-1)\ell}|^{-1}T_\omega^\ell\varphi + c_1\delta((p-1)\ell) \text{ var } |DT_\omega^\ell|^{-1}\varphi.$$

Recursively one obtains for all $\varphi \in \mathscr{I}_{p\ell}^\omega$:

$$\begin{aligned} \text{var } |DT_\omega^{p\ell}|^{-1}\varphi &\le \sum_{j=0}^{p-1}(c_1\delta(\ell))^j c_1\delta((p-j-1)\ell) \text{ var } |DT_\omega^\ell|^{-1}T_\omega^{j\ell}\varphi \\ &\le \sum_{j=0}^{p-1}(c_1\delta(\ell))^j c_1\delta((p-j-1)\ell) \text{ var } |DT_\omega^\ell|^{-1}. \end{aligned}$$

If we choose ℓ so that $\tilde{\eta} = c_1\delta(\ell) < 1$ then we obtain

$$\text{var } |DT_\omega^n|^{-1}\varphi \le \Delta(n) \text{ var } |DT_\omega^\ell|^{-1}$$

for all $\varphi \in \mathscr{I}_n^\omega$, for all $n \in \mathbb{N}$ and for some function $\Delta(n)$ which decays to zero at the same rate as $\delta(n)$.

The variation of the transfer operator is then estimated as:

$$\text{var}_I \mathcal{L}_\omega^n\psi = \text{var} \sum_{\varphi\in\mathscr{I}_n^\omega} (|DT_\omega^n|^{-1}\psi)\varphi \le \sum_{\varphi\in\mathscr{I}_n^\omega} \left(\text{var } \psi\varphi||DT_\omega^n|^{-1}\varphi|_\infty + |\psi\varphi|_\infty \text{ var } |DT_\omega^n|^{-1}\varphi\right).$$

Since $||DT_\omega^n|^{-1}|_\infty \le c_1|\zeta_\varphi| \le c_1\delta(n)$ one obtains

$$\text{var}_I \mathcal{L}_\omega^n\psi \le c_1\delta(n)\sum_\varphi \text{var } \psi\varphi + c_2\Delta(n)\sum_\varphi |\psi\varphi|_\infty$$

where $c_2 = \sup_\omega |DT_\omega^\ell|^{-1}$. With the estimate $|\psi\varphi|_\infty \le \int_I |\psi\varphi|\, d\lambda + \text{var } \psi\varphi$ this leads to

$$\text{var}_I \mathcal{L}_\omega^n\psi \le (c_1\delta(n) + c_2\Delta(n)) \text{ var } \psi + c_2\Delta(n)\left|\frac{1}{|DT_\omega^n|^{-1}}\right|_\infty \int_I \sum_\varphi |\psi\varphi|_\infty |DT_\omega^n|^{-1}\varphi\, d\lambda$$

as $\sum_\varphi \text{var}\ \psi\varphi = \text{var}\ \psi$. Since the Lebesgue measure λ is a fixed point of the transfer operator we finally get

$$\text{var}_I\ \mathcal{L}_\omega^n \psi \le (c_1\delta(n) + c_2\Delta(n))\,\text{var}\ \psi + c_2\Delta(n)|DT_\omega^n|_\infty \int_I |\psi|_\infty\, d\lambda.$$

Now, if we choose n so that $\eta = c_1\delta(n) + c_2\Delta(n) < 1$ we obtain

$$\text{var}_I\ \mathcal{L}_\omega^n\psi \le \eta\ \text{var}\ \psi + c_3\|\psi\|_{\mathscr{L}^1},$$

where $c_3 \ge c_2\Delta(n)|DT_\omega^n|_\infty$. Put $C_4 = c_3$. Note that the constant $\eta < 1$ can be chosen arbitrarily small by taking n sufficiently large. □

This proves the property (LY2) of [9]. The other two properties (LY0) and (LY1) are naturally satisfied as are the properties (V). To verify the random covering condition (RC) in [9] let $\psi \in \mathcal{C}_a$ where $\mathcal{C}_a = \{\psi > 0 : \text{var}\ \psi \le a\|\psi\|_{\mathscr{L}^1}\}$. Then by iterating Lemma 7.1 one obtains

$$\text{var}\ \mathcal{L}_\omega^m\psi \le \eta^{\frac{m}{n}}\ \text{var}\ \psi + \frac{C_4}{1-\eta^{\frac{1}{n}}}\|\psi\|_{\mathscr{L}^1} \le \left(\eta^{\frac{m}{n}} a + \frac{C_4}{1-\eta^{\frac{1}{n}}}\right)\|\psi\|_{\mathscr{L}^1}$$

for all large m. If we choose $a \ge \frac{4C_4}{(1-\eta^{\frac{1}{n}})^2}$ then for all m large enough $\eta^{\frac{m}{n}}a < \frac{1}{4}$ which then implies $\inf \mathcal{L}_\omega^m\psi \ge \|\mathcal{L}_\omega^m\psi\|_{\mathscr{L}^1} - \text{var}\ \mathcal{L}_\omega^m\psi \ge \frac{a}{2}\|\psi\|_{\mathscr{L}^1}$. Hence the condition (RC) of [9] is satisfied with $\alpha_n = \frac{a}{2}$.

Therefore by the Main Theorem of [9] there exists a family of absolutely continuous measures μ^ω on the fibres $\{\omega\}\times I$ which satisfy the generalised invariance property $T_\omega^*\mu^\omega = \mu^{\theta\omega}$. In particular there is an S-invariant measure $\mathbb{P}$ on $\Omega \times I$ which is of the form $d\mathbb{P}(\omega, x) = d\mu^\omega(x)d\nu(\omega)$, where ν is a θ-invariant measure on Ω. Also note as a consequence of the lower bound $\inf\psi \ge \frac{a}{2}\|\psi\|_{\mathscr{L}^1}$ the densities h_ω of μ^ω have a uniform lower bound, that is there exists a constant $c_1 > 0$ so that $\inf h_\omega \ge c_1$ for all $\omega \in \Omega$. Furthermore, the quenched decay of correlations holds at exponential rate.

To obtain the annealed decay, we assume that the underlying measure ν has decay of correlations for Hölder continuous functions and L^∞ functions, and show that the skew system has annealed decay of correlations. For this purpose, we put, for $\kappa \in (0,1)$,

$$C_\kappa(\Omega,\mathbb{R}) = \{\tilde{G} : \Omega \to \mathbb{R}\ : \|\tilde{G}\|_\kappa < \infty\},$$

where $\|\tilde{G}\|_\kappa = \|\tilde{G}\|_\infty + |\tilde{G}|_\kappa$, $|\tilde{G}|_\kappa = \sup_{n\ge n}\kappa^{-n}\,\text{var}_n\ \tilde{G}$ and $\text{var}_k\ \tilde{G} = \sup_{\omega,\omega'\in\Omega,\omega_i=\omega_i',|i|\le k}|\tilde{G}(\omega) - \tilde{G}(\omega')|$. Then $C_\kappa(\Omega,\mathbb{R})$ with the norm $\|\cdot\|_\kappa$ is a Banach space.

Lemma 7.2. *Suppose that the quenched decay of correlations is at rate $\lambda(k)$ and assume that the base system (Ω,θ,ν) has decay of correlations for Hölder continuous functions and L^∞ functions at the same rate, that is,*

$$|\nu(\tilde{G}\cdot(\tilde{H}\circ\theta^k)) - \nu(\tilde{G})\nu(\tilde{H})| \le \lambda(k)\|\tilde{G}\|_\kappa\|\tilde{H}\|_{L^\infty(\Omega,\mathbb{R})}$$

for $\tilde{G} \in C_\kappa(\Omega,\mathbb{R})$ (see below), $\tilde{H} \in L^\infty(\Omega,\mathbb{R})$.

Then the skew system $(\Omega\times M, S, \mu)$ has annealed decay of correlations for $G \in Lip(M,\mathbb{R})$, $H \in L^\infty(M,\mathbb{R})$ at rate $\lambda(k)$.

Proof. It follows from Lemma 7.1 by standard arguments that $\|\mathcal{L}_\omega^n \varphi\|$ decays exponentially uniformly in ω for functions φ which satisfy $\int \varphi\, dx = 0$, where $\|\cdot\|$ is the norm on M defined by the $\mathscr{L}^1$-norm plus the variation semi-norm.

Set $\varphi_\omega = 1 - h_\omega$. In particular $\int \varphi_\omega\, dx = 0$ and $\operatorname{var} \mathcal{L}_\omega^n \varphi_\omega \leq c_1 \kappa^n$ for some $c_1 > 0$, $\kappa \in (0,1)$ that do not depend on ω. In the supremum norm we thus get the following uniform bound

$$\|\mathcal{L}_\omega^n \varphi_\omega\|_\infty \leq \|\mathcal{L}_\omega^n \varphi_\omega\|_{\mathscr{L}^1} + \operatorname{var} \mathcal{L}_\omega^n \varphi_\omega \leq c_2 \kappa^n,$$

where $\|\cdot\|_\infty$ is the L^∞ norm on M. Apply the previous inequality on $\theta^{-n}\omega$ and $\theta^{-n}\omega'$ and note that $\mathcal{L}_{\theta^{-n}\omega}^n = \mathcal{L}_{\theta^{-n}\omega'}^n$ if $\omega_i = \omega_i'$ for $|i| \leq n$ as the map $\omega \to T_\omega$ depends by assumption only on the coordinate ω_0. In this case $\mathcal{L}_{\theta^{-n}\omega}^n h_{\theta^{-n}\omega} = h_\omega$ and by the triangle inequality

$$\|h_\omega - h_{\omega'}\|_\infty \leq 2c_2\kappa^n$$

for all ω, ω' with $\omega_i = \omega_i'$, $|i| \leq n$. In particular we get that for bounded functions G:

$$|\mu_\omega(G) - \mu_{\omega'}(G)| \leq 2c_2\kappa^n \|G\|_\infty \leq 2c_2\kappa^n\|G\|. \tag{9}$$

As a consequence we get a function $\tilde{G} : \Omega \to \mathbb{R}$ defined by $\tilde{G}(\omega) = \mu_\omega(G)$ which has exponentially decreasing variation:

$$\operatorname{var}_n \tilde{G} = \sup_{\omega_i = \omega_i', |i| \leq n} |\tilde{G}(\omega) - \tilde{G}(\omega')| \leq 2c_2\kappa^n \|G\|_\infty$$

which implies that the usual shift space Hölder semi-norm [8] is bounded:

$$|\tilde{G}|_\kappa = \sup_{n \geq 1} \kappa^{-n} \operatorname{var}_n \tilde{G} \leq 2c_2 \|G\|_\infty.$$

Moreover, we get

$$\|\tilde{G}\|_\kappa = \|\tilde{G}\|_{L^\infty(\Omega,\mathbb{R})} + |\tilde{G}|_\kappa \leq \|G\|_\infty + 2c_2\|G\|_\infty \leq c_3 \|G\|_\infty.$$

Now we take $G \in Lip(M,\mathbb{R})$ and $H \in L^\infty(M,\mathbb{R})$ and naturally consider them to be functions on $\Omega \times M$. The previous argument shows that $\tilde{G}(\omega) = \mu_\omega(G) \in C_\kappa(\Omega,\mathbb{R})$. Moreover, the function $\tilde{H}(\omega) = \mu_\omega(H) \in L^\infty(\Omega,\mathbb{R})$ as a function of ω, with $\|\tilde{H}\|_{L^\infty(\Omega,\mathbb{R})} \leq \|H\|_\infty$.

Then by the quenched decay of correlations and the decay of correlations on the shift space:

$$\begin{aligned}
\mu(G \cdot (H \circ S^k)) &= \int_\Omega \mu_\omega(G \cdot (H \circ T_\omega^k))\, d\nu(\omega) \\
&= \int_\Omega \mu_\omega(G) \cdot \mu_{\theta^k\omega}(H)\, d\nu(\omega) + \mathcal{O}(\lambda(k)\|G\|\|H\|_\infty) \\
&= \nu(\tilde{G})\nu(\tilde{H}) + \mathcal{O}(\lambda(k)\|\tilde{G}\|_\kappa \|\tilde{H}\|_{L^\infty(\Omega,\mathbb{R})}) + \mathcal{O}(\lambda(k)\|G\|\|H\|_\infty).
\end{aligned}$$

Since $\mu(G) = \nu(\tilde{G})$, $\mu(H) = \nu(\tilde{H})$ and $\|\tilde{G}\|_\kappa \leq c_3\|G\|_\infty \leq c_3\|G\|$ the statement of the lemma follows with decay rate $\lambda(k)$.

□

Remark 7.1. *By classical theory of equilibrium states on shift spaces [8], if ν is an equilibrium state for some Hölder potential $F \in C_\kappa(\Omega, \mathbb{R})$, $\kappa \in (0,1)$, then the correlations decay exponentially:*

$$|\nu(\tilde{G} \cdot (\tilde{H} \circ \theta^k)) - \nu(\tilde{G})\nu(\tilde{H})| \leq c_4 \tilde{\kappa}^k \|\tilde{G}\|_\kappa \|\tilde{H}\|_{L^\infty(\Omega,\mathbb{R})},$$

for some $\tilde{\kappa} \in (0,1)$, where $\|\tilde{G}\|_\kappa = \|\tilde{G}\|_{L^\infty(\Omega,\mathbb{R})} + |\tilde{G}|_\kappa$ is the norm of κ-Hölder continuous functions on the shift space Ω. Then the previous lemma shows that the skew system has annealed decay of correlations at exponential speed provided $\lambda(k)$ decays exponentially.

Alternatively one could have a measure ν with slower rate of decay (say, superpolynomially) and Lemma 7.2 gives the annealed decay at the same rate (provided $\lambda(k)$ decays at least as fast).

Since the measures μ^ω are absolutely continuous with respect to Lebesgue measure, condition (V) is satisfied with any values $d_0 < 1 < d_1$ arbitrarily close to 1 and from the uniform lower bound on the densities h_ω, i.e. we can take $K = 1/c_1$. Condition (III) follows from the uniform boundedness of second order derivatives. The annulus condition (VI) is satisfied with $\xi = \beta = 1$. We can therefore invoke Theorems 2.1 and 2.2 and obtain the following result:

Theorem 7.1. *Let $S : \Omega \times I \circlearrowleft$ be a skew system as described above, where the maps T_ω are piecewise C^2 with uniformly bounded C^2 derivatives. Let $\delta(n)$ be a summable sequence which monotonically decreases to zero so that $|\zeta_\varphi| \leq \delta(n)$ for all $\varphi \in \mathscr{I}_n^\omega$ for all n. Then*

$$\mu^\omega\left(y \in [0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

and

$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y \in [0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

for all $t > 0$, for Lebesgue almost every $\mathsf{x} \in [0,1]$ and ν-almost every $\omega \in \Omega$.

Clearly, if the maps T_ω are uniformly expanding then δ decays exponentially and satisfies the requirement of the theorem.

To our knowledge, this is the first result on the hitting time distribution for random systems where the base measure ν is not i.i.d..

7.2. **Random parabolic interval maps.** We use the family of Pomeau-Manneville maps indexed by $\alpha > 0$ which is given by

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}) \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

These maps have a neutral (parabolic) fixed point at $x = 0$ and are otherwise expanding. It is known that if $\alpha < 1$ then there exists an invariant absolutely continuous probability measure. Here we assume the setting of [7] (see [6] for the non-invertible setting). Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the 'driving space' with the left shift map $\theta : \Omega \circlearrowleft$. We equip Ω with the

Bernoulli measure ν with weights $(\frac{1}{2}, \frac{1}{2})$. Let $0 < \alpha_0 < \alpha_1 < 1$ and define the function $\alpha : \Omega \to \{\alpha_0, \alpha_1\}$ by

$$\alpha(\omega) = \begin{cases} \alpha_0 & \text{if } \omega_0 = 0 \\ \alpha_1 & \text{if } \omega_0 = 1 \end{cases}.$$

Then we have a skew action $S : \Omega \times I$, with $I = [0, 1]$ defined by $S(\omega, x) = (\theta(\omega), T_\omega x)$, where we wrote $T_\omega = T_{\alpha(\omega)}$. Its iterates are $S(\omega, x) = (\theta^n \omega, T^n_\omega x)$, where $T^n_\omega = T_{\theta^{n-1}\omega} \circ \cdots \circ T_{\theta\omega} \circ T_\omega$. It is shown in [7] that there exists an S-invariant probability measure $d\mu = h_\omega d\lambda d\nu(\omega)$, where λ is the Lebesgue measure on I and where the density h_ω is defined on ν-almost every ω and is Lipschitz continuous.

For the transfer operator $\mathcal{P}_\omega$ (adjoint to T_ω), one has $\mathcal{P}_\omega h_\omega = h_{\theta\omega}$, $\mathcal{P}^*_\omega \lambda = \lambda$ and also by [2] Theorem 1.6:

$$\begin{aligned} & \left| \int \psi(\phi \circ T^n_\omega)\, d\mu^\omega - \int \psi\, d\mu^\omega \int \phi\, d\mu^{\theta^n \omega} \right| \\ = & \left| \int (\mathcal{P}^n_\omega \psi h_\omega) \phi\, d\lambda - \mu^\omega(\psi) \int \phi \mathcal{P}^n_\omega h_\omega\, d\lambda \right| \\ \leq & \int |\phi| \cdot |\mathcal{P}^n_\omega(\psi h_\omega - h_\omega \mu^\omega(\psi))|\, d\lambda \\ \leq & \; c_1 |\phi|_\infty \left(\|\psi h_\omega\|_{\mathscr{L}^1(\lambda)} + \|h_\omega \mu^\omega(\psi)\|_{\mathscr{L}^1(\lambda)} \right) \frac{\log^{\frac{1}{\alpha_1}} n}{n^{\frac{1}{\alpha_1}-1}}, \end{aligned}$$

for some constant c_1 (which by [2] is equal to $\max\{C_{\alpha_0}, C_{\alpha_1}\}$). This is under the stated assumption that ϕh_ω and $h_\omega \mu^\omega(\psi)$ belong to the cone of functions $\mathcal{C}_2$ which is defined as

$$\mathcal{C}_2 = \left\{ f \in C^0((0, 1]) \cap \mathscr{L}^1(\lambda) : f \geq 0, f \text{ decreasing}, x^{1+\alpha} f \text{ increasing}, f(x) \leq a x^{-\alpha} \lambda(f) \right\}.$$

This in fact applies to the function $h_\omega \mu^\omega(\psi)$. A careful reading of the proof makes it apparent that the class of functions to which the contraction applies is far wider and in fact is only determined by the property that $\|\phi - \mathbb{A}_\epsilon \phi\|_{\mathscr{L}^1}$ is bounded by a multiple of $\epsilon^{1-\alpha}$. The smoothing operator $\mathbb{A}_\epsilon$ is given by $\mathbb{A}_\epsilon \phi(x) = \frac{1}{2\epsilon} \int_{B_\epsilon(x)} \phi(y)\, d\lambda(y)$. Since we want ψ to be the indicator function of B_ρ, this requirement is clearly satisfied as $\|\phi - \mathbb{A}_\epsilon \phi\|_{\mathscr{L}^1} \lesssim \epsilon$. Consequently, for the purposes of Theorem 2.1, Assumption (I) is satisfied with $\lambda(n) = \mathcal{O}(n^{-p})$ for any $p < \frac{1}{\alpha_1} - 1$. Since one can integrate w.r.t. $d\nu(\omega)$, Assumption (II) is satisfied with the same λ.

Clearly the dimension of μ is equal to one and Assumption (V) is satisfied with any $d_0 < 1 < d_1$ arbitrarily close to 1 from the fact that the density functions h_ω are uniformly bounded and bounded away from 0. Assumption (VI) is satisfied with $\xi = \beta = 1$. Also, if we denote by $\psi^n_{\theta^{-n}\omega}$ the (unique) inverse branch of $T^n_{\theta^{-n}\omega}$ which contains the parabolic point 0, then one has that $|\psi^n_{\theta^{-n}\omega}(I)| = \mathcal{O}(n^{-1/\alpha_1})$ for all ω. Hence $\delta(n) = \mathcal{O}(n^{-\kappa})$ with $\kappa = 1/\alpha_1$.

To estimate the distortion we again look at the 'worst case' which are the parabolic inverse branches $\psi^n_{\theta^{-n}\omega}$ of the map $T^n_{\theta^{-n}\omega}$. Put $a_n(\omega) = \psi^n_{\theta^{-n}\omega}(a_0)$, where $a_0 = \frac{1}{2}$. Then

$$DT^n_{\theta^{-n}\omega}(a_n) = \prod_{j=1}^{n} DT_{\theta^{-j}\omega}(a_j).$$

For the parabolic branch: $DT_{\theta^{-j}\omega}(x) = 1 + (1+\alpha(\theta^{-j}\omega))2^{1+\alpha(\theta^{-j}\omega)}x^{\alpha(\theta^{-j}\omega)})$. Also

$$a_{j-1} = T_{\theta^{-j}\omega}(a_j) = a_j\left(1 + 2^{1+\alpha(\theta^{-j}\omega)}a_j^{\alpha(\theta^{-j}\omega)}\right) = a_j(DT_{\theta^{-j}\omega}(a_j))^{1/(1+\alpha(\theta^{-j}\omega)} + \text{ h.o.t.}$$

and

$$DT_{\theta^{-j}\omega}(a_j) = \left(\frac{a_{j-1}}{a_j}\right)^{1+\alpha(\theta^{-j}\omega)} + \text{ h.o.t.}$$

which implies the estimate

$$DT^n_{\theta^{-n}\omega}(a_n) \le c_2 \prod_{j=0}^{n-1}\left(\frac{a_{j-1}}{a_j}\right)^{1+\alpha(\theta^{-j}\omega)} \le c_2\left(\prod_{j=0}^{n-1}\frac{a_{j-1}}{a_j}\right)^{1+\alpha_1} = c_2\, a_n^{-(1+\alpha_1)}.$$

Denote by $a'_k = a_k(\alpha_0)$ the inverse images of $a_0 = \frac{1}{2}$ in the deterministic case when all maps have the parameter value α_0. Then $a'_k \sim c_3 k^{-1/\alpha_0}$ for some $c_3 > 0$. Let us put $A_k = (a'_{k-1}, a_0]$. In order to estimate the distortion of the maps T^n_ω on the images of A_k and $A = (a_0, 1]$ under the inverse branches $\mathscr{I}^\omega_n$ of T^n_ω we look at the 'worst case' when the inverse branch is the parabolic branch ψ^n_ω. Then

$$\Theta(n) = \frac{DT^n_{\theta^n\omega}(a_n(\omega))}{DT^n_{\theta^{n+k'}\omega}(\tilde{a}_{n+k'}(\omega))}$$

where $\tilde{a}_{n+k'} = \psi^n_{\theta^{n+k'}\omega}(a'_k)$ and k' is so that $a'_k \in \psi^{k'}_{\theta^{k'}\omega}A$. We estimate the numerator from above by

$$DT^n_{\theta^n\omega}(a_n(\omega)) = \mathcal{O}(1)\prod_{j=0}^{n-1}\left(\frac{a_j}{a_{j+1}}\right)^{1+\alpha(\theta^{-j}\omega)} \lesssim \left(\frac{a_0}{a_n}\right)^{1+\alpha_1} \lesssim n^{\frac{1+\alpha_1}{\alpha_0}}$$

as $a_n \ge a'_n = \mathcal{O}(n^{-1/\alpha_0})$. The denominator is estimated from below as follows:

$$DT^n_{\theta^{n+k'}\omega}(\tilde{a}_{n+k'}(\omega)) = \mathcal{O}(1)\prod_{j=k'}^{n+k'-1}\left(\frac{\tilde{a}_j}{\tilde{a}_{j+1}}\right)^{1+\alpha(\theta^{-j}\omega)} \gtrsim \left(\frac{\tilde{a}_{k'}}{\tilde{a}_{n+k'}}\right)^{1+\alpha_0}.$$

Since $|\tilde{a}_{j+k'} - \tilde{a}_{j-1+k'}| = 2^{1+\alpha(\theta^{-j}\omega)}\tilde{a}_{j+k'}^{1+\alpha(\theta^{-j}\omega)} \le 2^{1+\alpha_1}\tilde{a}_{j+k'}^{1+\alpha_1}$ one obtains

$$\frac{\tilde{a}_{k'}}{\tilde{a}_{n+k'}} \gtrsim \frac{\tilde{a}_{k'}}{\left(\tilde{a}_{k'}^{-\alpha_1} + n\right)^{-1/\alpha_1}}$$

where $\tilde{a}_{k'} \sim c_3 n^{-\eta/\alpha_0}$ for some $\eta > 0$. Hence

$$\Theta(n) \lesssim n^{\frac{1+\alpha_1}{\alpha_0}}\left(n^{\eta\frac{\alpha_1}{\alpha_0}} + n\right)^{-\frac{1+\alpha_0}{\alpha_1}} n^{\eta\frac{1+\alpha_0}{\alpha_0}}.$$

If we choose $\eta > 0$ so that $\eta\frac{\alpha_1}{\alpha_0} < 1$ then these two estimates combined yield

$$\Theta(n) \le c_4 n^{\kappa'},$$

where $\kappa' = \frac{1+\alpha_1}{\alpha_0} - \frac{1+\alpha_0}{\alpha_1} + \eta\frac{1+\alpha_0}{\alpha_0}$.

Assuming that $0 < \alpha_0 < \alpha_1 < \frac{1}{3}$, the condition $u_0\kappa - 2 - \kappa' > 1$ of Theorem 2.1 is satisfied since

$$\kappa - 2 - \kappa' = \frac{1}{\alpha_0\alpha_1}\big(\alpha_1^2 - \alpha_0^2 + \alpha_1(1+\eta-2\alpha_0+\eta\alpha_0)\big) > \frac{(1/3+\eta)}{\alpha_0} > 1$$

is positive for any $\eta > 0$ and u_0 can be chosen arbitrarily close to 1.

For the inverse branches $\mathscr{I}_n^\omega$ of T_ω^n denote by $\hat{\zeta}_\varphi$ the 'n-cylinder' that is a pre-image of either $A_k \cup A = (a'_{k-1}, 1]$ under the inverse branches $\varphi \in \mathscr{I}_n^\omega$. We can now nearly use Theorem 2.1. Note that if $\mathsf{x} \in (0,1)$ then for all n large enough $\mathsf{x} \in A_{n^\theta} \cup A$. If we proceed as in the estimate of the term $\mathcal{R}_2$ in Section 4.2 we obtain

$$T_\omega^{-j} B_\rho \cap B_\rho \subset \bigcup_{\zeta:\zeta\cap B_\rho \neq \varnothing} T^{-j} B_\rho \cap \zeta = \mathscr{P}_1 \cup \mathscr{P}_2$$

where the union is over j-cylinders ζ and

$$\mathscr{P}_1 = \bigcup_{\zeta:\zeta\cap B_\rho\neq\varnothing} T_\omega^{-j} B_\rho \cap \hat{\zeta}, \qquad \mathscr{P}_2 = \bigcup_{\zeta:\zeta\cap B_\rho\neq\varnothing} T_\omega^{-j} B_\rho \cap \zeta \setminus \hat{\zeta}.$$

The first set is estimated as before in the main theorem. For the second term notice that

$$\mathscr{P}_2 = \bigcup_{\varphi\in\mathscr{I}_j^\omega:\varphi(I)\cap B_\rho\neq\varnothing} T_\omega^{-j} B_\rho \cap \varphi(D_{n^\eta})$$

where $D_k = (0, a'_{k-1}]$. Hence

$$\mathscr{P}_2 = \bigcup_{\varphi\in\mathscr{I}_j^\omega:\varphi(I)\cap B_\rho\neq\varnothing} \varphi(B_\rho \cap D_{n^\eta})$$

is empty for n large enough, i.e. so that $a_{n^\eta} < \mathsf{x}$.

Theorem 7.2. *Let $S : \{0,1\}^{\mathbb{Z}} \times I \circlearrowleft$ be the random system described above, where the maps T_ω are the parabolic maps T_{α_0} and T_{α_1}. Assume $0 < \alpha_0 < \alpha_1 < \frac{1}{3}$. Denote by μ the annealed invariant measure and by μ^ω the fibred measures. Then for all $t > 0$:*

$$\mu^\omega\left(y \in [0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t}$$
$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y \in [0,1] : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t}$$

as $\rho \to 0$ for Lebesgue almost every $\mathsf{x} \in [0,1]$ and ν-almost every $\omega \in \{0,1\}^{\mathbb{Z}}$.

Proof. We verify the conditions of the theorem. Above we verified Assumptions (I)–(VI). Otherwise, as $\xi = 1$ and $\alpha_1 < \frac{1}{3}$ one clearly has $\kappa\xi > 1$. Also, since u_0 and d_1 can be chosen arbitrarily close to one, $\beta = 1$ and choosing $p = \frac{1}{\alpha'_1} - 1$ with $\frac{1}{3} > \alpha'_1 > \alpha_1$ arbitrarily close to α_1 we get $\max\left(\frac{d_1\beta}{\kappa\xi-1}, (\frac{\beta}{\xi}+d_1)\frac{1}{p}\right) \leq \max(\frac{d_1}{2}, (1+d_1)\frac{\alpha'_1}{1-\alpha'_1}) < \min(1, u_0)$.

□

7.3. **Random perturbation of partially hyperbolic attractors.** In [4] the authors consider the small random perturbation of certain partially hyperbolic attractors. To be more precise, they consider a neighborhood $\mathcal{F}$ of a $C^{1+\alpha}$ diffeomorphism $f \in \mathrm{Diff}^{1+\alpha}(M)$ on a compact Riemannian manifold M with dimension at least two. Given any Borel probability measure β whose support is a compact subset $B \subset \mathcal{F}$, they consider the shift space $\Omega = B^{\mathbb{Z}}$ with invariant probability $\nu = \beta^{\mathbb{Z}}$. Then the skew product $(\omega, x) \to (\theta\omega, f_\omega(x))$ is a random dynamical system where the map f_ω only depends on the first symbol of ω.

Then it is assumed that there exists a *hyperbolic product structure* Λ_ω, consisting of continuous families of stable and unstable leaves (all of which depend on ω), such that

- Λ_ω has positive Lebesgue volume on each unstable leaf;
- there is a measurable partition $\{\Lambda_{i,\omega}\}$ of Λ_ω into u-subsets and a return time function $R_{i,\omega}$, constant on each $\Lambda_{i,\omega}$, such that the return map is Markov in the sense that
$$f_\omega^{R_{i,\omega}}(\gamma_\omega^u(x)) \supset \gamma^u_{\theta^{R_{i,\omega}}\omega}(f_\omega^{R_{i,\omega}}(x)),$$
and a similar relation holds for the stable leaves.
- f_ω^n contracts the stable leaves exponentially fast;
- on the unstable leaves, the return map $f_\omega^{-R_\omega}$ contracts exponentially fast, with bounded distortion that is independent of ω;
- the stable holonomy is absolute continuous, with a log-Hölder density whose Hölder constant is independent of ω.

Such conditions are similar to Assumption (III) and (IV) in Theorem 2.1, and can be seen as the random version of the invertible Young's tower in [27]. Under the assumption that the tail of the return time is uniformly summable:
$$\sum_{n\ge 0} \mathrm{Leb}_{\gamma^u_{\theta^{-n}\omega}}\{R_{\theta^{-n}\omega} > n\} \le C$$
for some constant C that is independent of ω, it is proven that the system admits a family of physical measures $\{\mu^\omega\}$ with $(f_\omega)_*\mu^\omega = \mu^{\theta\omega}$. More importantly, μ^ω have absolutely continuous conditional measures along the unstable leaves, whose density w.r.t. the leaf volume is uniformly bounded above and below. Furthermore, if one assumes that the tail of the return time, $\{R_{\theta^{-n}\omega} > n\}$, is exponential/stretch exponential/polynomially small, then the system has quenched and annealed decay of correlations for Hölder continuous functions at corresponding rates.

As a specific example, we take $f \in \mathrm{Diff}^{1+\alpha}(M)$ where M is a compact Riemannian manifold, such that f has a topologically mixing uniformly hyperbolic attractor $K \in M$. Assume that $\{f_\omega\}_{\omega\in\Omega}$ is a small random perturbation of f. Then it is proven in [4, Theorem 1.6] that the system has quenched decay of correlations for Hölder functions G, H at exponential speed, with constant $C_{G,H}$ independent of ω. Furthermore, it can be seen from the proof that H is allowed to be L^∞ on local stable leaves.[6]

[6]For deterministic Young towers, it is a well known fact that the decay of correlations holds for Hölder functions against L^∞ functions that are constant on local stable leaves; the proof of [4] follows the argument of [27], and the same conclusion holds.

Due to the uniform exponential contracting/expanding on stable and unstable leaves, Assumption (III) and (IV) naturally holds with $\Theta(n)$ being a constant, and $\delta(n)$ decaying exponentially fast. Assumption (V) holds since the density of the conditional measures of μ^ω are equivalent to the leaf volume with uniformly bounded density, and γ^u_ω depends continuously on ω. In the mean time, it is known that Assumption (VI) holds for the physical measure of (the unperturbed map) f, see for example [15]. Note that the proof in [15] only uses the fact that the conditional measures are absolutely continuous w.r.t. the Lebesgue measure; as a result, a similar proof shows that (VI) holds for the physical measures $\{\mu^\omega\}$.

Therefore we obtain the following theorem:

Theorem 7.3. *Let $\{f_\omega\}$ be a family of same C^1 perturbations of $f \in \mathit{Diff}^{1+\alpha}(M)$ which has a topologically mixing, uniformly hyperbolic attractor $K \subset M$. Then the random system $S(\omega, x) = (\theta\omega, f_\omega(x))$ and the physical measures $\{\mu^\omega\}$ satisfy*

$$\mu^\omega\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

and

$$\mu^\omega_{B_\rho(\mathsf{x})}\left(y \in M : \tau^\omega_{B_\rho(\mathsf{x})}(y) > \frac{t}{\mu(B_\rho(\mathsf{x}))}\right) \longrightarrow e^{-t} \qquad \text{as } \rho \to 0$$

for all $t > 0$, for μ^ω-almost every $\mathsf{x} \in M$ and ν-almost every $\omega \in \Omega$.

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